Evaluate The Iterated Integral. 6 1 X 0 (5x 2y) Dy Dx

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Understanding how to evaluate iterated integrals is a fundamental skill in multivariable calculus. These integrals allow us to compute the volume under surfaces, area over regions, and various other quantities involving functions of two or more variables. In this article, we will delve into the detailed process of evaluating the specific iterated integral:

$$
\int_{x=1}^{6} \int_{y=0}^{5x} (5x + 2y) \, dy \, dx
$$

By exploring this integral step-by-step, you'll gain a comprehensive understanding of the techniques involved, including setting up the limits, integrating with respect to the inner variable, and finally integrating with respect to the outer variable.

Understanding the Structure of the Integral

Before jumping into calculations, it’s crucial to interpret the integral properly. The given expression:

$$
\int_{x=1}^{6} \int_{y=0}^{5x} (5x + 2y) \, dy \, dx
$$

represents a double integral with the inner integral integrating over y from 0 to $5x$, and the outer integral integrating over x from 1 to 6.

Key points to note:

- The integrand $5x + 2y$ is a function of both x and y.
- The limits for y depend on x, indicating a region that is bounded by the lines $y=0$ and $y=5x$.
- The outer limits for x are constants, from 1 to 6.

This structure suggests that the region of integration is bounded by the lines:

- x from 1 to 6,
- y from 0 up to $5x$.

Visualizing this region in the xy-plane helps clarify the bounds and the nature of the integral.

Visualizing the Region of Integration

Understanding the region over which we integrate is essential. The bounds:

- x runs from 1 to 6,
- For each fixed x, y runs from 0 to $5x$.

Plotting the boundary lines:

- The line $y = 0$ is the x-axis.
- The line $y = 5x$ passes through the origin with a slope of 5, but since x starts at 1, the region begins at $(1, 0)$ and extends to $(6, 30)$.

This region forms a curvilinear trapezoid bounded by:

- The vertical lines $x=1$ and $x=6$,
- The curve $y=5x$,
- The horizontal line $y=0$.

Step-by-Step Evaluation of the Inner Integral

The first step in evaluating the double integral is to compute the inner integral with respect to y.

Inner integral:

$$
I(x) = \int_{0}^{5x} (5x + 2y) \, dy
$$

Since $5x$ is treated as a constant with respect to y, we can split the integral:

$$
I(x) = \int_{0}^{5x} 5x \, dy + \int_{0}^{5x} 2y \, dy
$$

Let's evaluate each term separately.

1. Integral of $5x$ with respect to y:

$$
\int_{0}^{5x} 5x \, dy = 5x \times (5x - 0) = 5x \times 5x = 25x^2
$$

2. Integral of $2y$ with respect to y:

$$
\int_{0}^{5x} 2y \, dy = \left[y^2 \right]_0^{5x} = (5x)^2 - 0 = 25x^2
$$

Sum of the two parts:

\[
I(x) = 25x^2 + 25x^2 = 50x^2
\]

Thus, the inner integral simplifies to:

\[
I(x) = 50x^2
\]

Evaluating the Outer Integral

Having computed the inner integral, the original double integral reduces to a single integral:

\[
\int_{1}^{6} I(x) \, dx = \int_{1}^{6} 50x^2 \, dx
\]

Now, we will evaluate this integral.

Step-by-step:

\[
\int_{1}^{6} 50x^2 \, dx = 50 \int_{1}^{6} x^2 \, dx
\]

The integral of \(x^2 \) with respect to \(x \):

\[
\int x^2 \, dx = \frac{x^3}{3}
\]

Applying the limits:

\[
50 \times \left[\frac{x^3}{3} \right]_1^6 = \frac{50}{3} \left(6^3 - 1^3
\right)
\]

Calculate:

\[
6^3 = 216, \quad 1^3 = 1
\]

Subtract:

\[
216 - 1 = 215
\]

Multiply:

\[

$$
\frac{50}{3} \times 215 = \frac{50 \times 215}{3} = \frac{10750}{3}
$$

Final Result and Interpretation

The value of the original iterated integral is:

$$
\boxed{\frac{10750}{3}}
$$

which is approximately 3583.33 when expressed as a decimal.

This result represents the accumulated quantity over the region described, such as volume under the surface $z = 5x + 2y$ over the specified bounds.

Summary of Steps for Evaluating Similar Integrals

To successfully evaluate an iterated integral similar to the one discussed, follow these guidelines:

1. Interpret the bounds:

- Understand whether the limits are constants or functions of other variables.
- Visualize the region to clarify the bounds.

2. Integrate the inner integral:

- Treat the outer variables as constants.
- Break the integral into manageable parts if necessary.

3. Simplify the result:

- After computing the inner integral, simplify the expression before proceeding.

4. Integrate the outer integral:

- Use standard integration techniques.
- Carefully evaluate the definite integral with the limits.

5. Verify the region and the calculations:

- Double-check the bounds and the integration steps.
- Confirm the calculations to avoid errors.

Additional Tips for Evaluating Iterated Integrals

- Change the order of integration if the region is complicated; sometimes integrating with respect to the other variable simplifies the process.
- Sketch the region for complex bounds to visualize the limits and ensure correctness.
- Use symmetry when possible to simplify calculations.
- Break down complex integrands into simpler parts and integrate term by term.

Conclusion

Evaluating the integral:

$$
\int_{x=1}^{6} \int_{y=0}^{5x} (5x + 2y) \, dy \, dx
$$

demonstrates the systematic approach required in multivariable calculus. By carefully analyzing the bounds, visualizing the region, and performing step-by-step integrations, you can accurately compute such integrals. Mastery of these techniques is crucial for solving real-world problems involving areas, volumes, and other quantities in multiple dimensions.

Whether you're tackling problems in physics, engineering, or mathematics, understanding how to evaluate iterated integrals will significantly enhance your problem-solving toolkit. Keep practicing with different types of bounds and integrands to develop confidence and proficiency in this essential area of calculus.

Frequently Asked Questions

How do you set up the iterated integral for evaluating the double integral $\int_0^1 \int_0^{6x} (5x^2 y)\, dy\, dx$?

The integral is set up with the outer integral over x from 0 to 1, and the inner integral over y from 0 to 6x, integrating the function $5x^2 y$ with respect to y first, then x.

What is the step-by-step process to evaluate the integral $\int_0^1 \int_0^{6x} (5x^2 y)\, dy\, dx$?

First, integrate the inner integral with respect to y: $\int_0^{6x} 5x^2 y\, dy = 5x^2 (y^2/2)$ evaluated from 0 to 6x, resulting in $5x^2 (36x^2/2) = 5x^2\ 18x^2 = 90x^4$. Then, integrate the outer integral over x from 0 to 1: $\int_0^1 90x^4\, dx$, which equals $90 (x^5/5)$ from 0 to 1 = $90 (1/5) = 18$.

What is the value of the double integral $\int_0^1 \int_0^{6x} (5x^2 y)$ dy dx?

The value of the integral is 18.

Can the order of integration be changed in this integral, and if so, how would the limits change?

Yes, the order can be changed. The region defined by $0 \leq x \leq 1$ and $0 \leq y \leq 6x$ can be described as $0 \leq y \leq 6$, with x from y/6 to 1. This would change the integral to $\int_0^6 \int_{y/6}^1 5x^2 y \, dx \, dy$.

What are common mistakes to avoid when evaluating this type of iterated integral?

Common mistakes include misreading the limits of integration, forgetting to include the substitution for the inner integral, and neglecting to properly evaluate the definite integrals step-by-step. Ensuring correct order and limits is crucial for accurate results.

Additional Resources

Evaluate The Iterated Integral. 6 1 X 0 (5x 2y) Dy Dx

When approaching the evaluation of the iterated integral $\int_{x=1}^{6} \int_{y=0}^{x} (5x + 2y) \, dy \, dx$, it is essential to understand both the structure of the integral and the methods used for its computation. This integral appears straightforward at first glance but reveals layers of complexity upon closer examination. Its evaluation not only demonstrates fundamental techniques in double integration but also emphasizes the importance of understanding the order of integration, bounds, and the integrand's nature. This article aims to provide a comprehensive review of the process involved, step-by-step calculations, common pitfalls, and the broader significance of such integrals in mathematical analysis.

Understanding the Integral's Structure

1. The Components of the Integral

The integral:

$$
\int_{x=1}^{6} \left(\int_{y=0}^{x} (5x + 2y) \, dy \right) dx
$$

is a double integral with variable limits on the inner integral. It is an iterated integral, meaning it integrates a function over a two-dimensional region by integrating first with respect to y (inner integral), then with respect to x (outer integral).

Key features:
- Inner integral bounds: $y = 0$ to $y = x$
- Outer integral bounds: $x = 1$ to $x = 6$
- Integrand: $5x + 2y$, a linear function in both variables

2. Visualization of the Region of Integration

Before performing calculations, visualizing the region helps clarify the limits. The bounds suggest:

- For each fixed x between 1 and 6
- y varies from 0 up to $y = x$

This describes a region in the xy-plane bounded by:

- The line $y = 0$ (the x-axis)
- The line $y = x$
- The vertical lines $x = 1$ and $x = 6$

Plotting this region confirms it is a triangular area between these lines, with vertices at points $(1, 0)$, $(6, 0)$, and $(6, 6)$.

Step-by-Step Evaluation of the Integral

1. Integrate with respect to y first

The inner integral:

$$
\int_{0}^{x} (5x + 2y) \, dy
$$

Since $5x$ is constant with respect to y, and $2y$ is straightforward, the integral becomes:

$$
\int_{0}^{x} 5x \, dy + \int_{0}^{x} 2y \, dy
$$

Calculating each:

- $\int_{0}^{x} 5x \, dy = 5x \int_{0}^{x} dy = 5x \cdot y \big|_{0}^{x} = 5x \cdot x = 5x^2$

- $\int_{0}^{x} 2y \, dy = 2 \cdot \frac{y^2}{2} \big|_{0}^{x} = y^2 \big|_{0}^{x} = x^2$

Adding these results:

$$
5x^2 + x^2 = 6x^2
$$

Thus, the inner integral simplifies to:

$$
\int_{0}^{x} (5x + 2y) \, dy = 6x^2
$$

2. Integrate with respect to x

Now, the original double integral reduces to:

$$
\int_{1}^{6} 6x^2 \, dx
$$

Calculating:

$$
6 \int_{1}^{6} x^2 \, dx
$$

The integral:

$$
\int x^2 \, dx = \frac{x^3}{3}
$$

Evaluated from 1 to 6:

$$
\frac{6^3}{3} - \frac{1^3}{3} = \frac{216}{3} - \frac{1}{3} = 72 - \frac{1}{3} = \frac{216}{3} - \frac{1}{3} = \frac{215}{3}
$$

Multiplying by 6:

$$
6 \times \frac{215}{3} = 2 \times 215 = 430
$$

Final answer:

$$
\boxed{430}
$$

Features and Insights of the Evaluation Process

Advantages of the Approach

- Simplicity: Direct integration after reducing the double integral to a single integral simplifies calculations.
- Clarity: Visualizing the region confirms the correctness of bounds and

integration order.
- Applicability: Demonstrates standard methods useful for more complex integrals with variable bounds.

Potential Challenges and Pitfalls

- Misinterpreting bounds: Incorrectly setting the limits can lead to errors.
- Miscomputing the inner integral: Forgetting to treat $5x$ as a constant with respect to y can cause mistakes.
- Order of integration: Sometimes reversing the order simplifies the evaluation, but it requires re-deriving bounds.

Alternative Approaches and Variations

Reversing the Order of Integration

In some cases, swapping the order of integration clarifies the integral, especially if the bounds are complicated. For this particular region, reversing the order involves:

- y ranging from 0 to 6
- For each y, x varies from $x = y$ (since $y \leq x$) to $x = 6$

The region can be described as:

$$
\int_{y=0}^{6} \int_{x=y}^{6} (5x + 2y) \, dx \, dy
$$

Calculating this integral involves:

- Inner integral:

$$
\int_{x=y}^{6} (5x + 2y) \, dx
$$

- Outer integral:

$$
\int_{0}^{6} (\text{result of inner integral}) \, dy
$$

This approach may sometimes simplify calculations, especially if integrand or bounds are more manageable in the reversed order.

Broader Context and Applications

1. Significance in Multivariable Calculus

Double integrals like this are fundamental in calculating areas, volumes, and accumulated quantities over regions in the plane. They are crucial in physics, engineering, and probability theory.

2. Extending to More Complex Regions and Integrands

Real-world problems often involve more intricate bounds and non-linear integrands. Mastering the evaluation of basic integrals like this builds a foundation for tackling such complexities.

3. Numerical Approximation and Computational Tools

When integrals become too complicated for analytical solutions, numerical methods such as Simpson's rule or Monte Carlo integration are employed. Familiarity with exact evaluation methods informs the development and application of these techniques.

Summary and Final Remarks

The evaluation of the iterated integral $\int_{x=1}^{6} \int_{y=0}^{x} (5x + 2y) \, dy \, dx$ exemplifies core principles of multivariable calculus. Key steps involve understanding the region of integration, carefully computing the inner integral, and then performing the outer integral. The process underscores the importance of visualization, correct treatment of bounds, and systematic calculation. The final result, 430, not only provides a numerical answer but also reinforces the conceptual clarity needed for more advanced integrations. Whether for academic purposes or practical applications, mastering such integrals enhances mathematical proficiency and prepares one for tackling real-world problems involving multivariate functions and regions.

Pros:
- Clear step-by-step methodology
- Demonstrates integral evaluation techniques
- Reinforces understanding of regions and bounds

Cons:
- Assumes familiarity with integral calculus
- Might be simplified or complicated depending on bounds and integrand
- Reversal of limits can sometimes be confusing without proper visualization

In conclusion, evaluating iterated integrals like this is a foundational skill in calculus, providing insight into the behavior of functions over regions and laying groundwork for advanced mathematical analysis.

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