Find The Derivative Of The Function Y=(x^2 4/x^2-4)^5

Find The Derivative Of The Function Y=(x^2 4/x^2-4)^5 is a calculus problem that involves differentiating a composite function with a quotient inside a power. This type of problem is common in calculus coursework and requires a good understanding of the chain rule, quotient rule, and algebraic simplification. In this article, we will explore step-by-step how to compute the derivative of this function, understand the underlying principles, and discuss related concepts to enhance your calculus skills.

Understanding the Given Function

Before delving into the differentiation process, it's essential to comprehend the structure of the function:

Y = (x^2 4 / (x^2 - 4))^5

This expression involves:

- A numerator: x^2 4
- A denominator: x^2 - 4
- An entire fraction raised to the 5th power

The function can be viewed as a composition of functions, which suggests that the chain rule will be integral to differentiating it.

Step 1: Simplify the Function (Optional but Recommended)

Although not strictly necessary, simplifying the function can make differentiation easier. Let's rewrite it:

Y = [(4x^2) / (x^2 - 4)]^5

Notice that the numerator is 4x^2, which can be factored out or kept as is. The main idea is to recognize the inner function:

u = (4x^2) / (x^2 - 4)

So, Y = u^5

Thus, the derivative dY/dx can be found using the chain rule:

$dY/dx = 5u^4 \, du/dx$

Our task reduces to finding du/dx.

Step 2: Find the Derivative of the Inner Function $u = (4x^2) / (x^2 - 4)$

Since u is a quotient, we'll apply the quotient rule:

If $u = f(x) / g(x)$, then

$du/dx = (f'(x) g(x) - f(x) g'(x)) / [g(x)]^2$

Identify:

- $f(x) = 4x^2$
- $g(x) = x^2 - 4$

Calculate derivatives:

- $f'(x) = 8x$
- $g'(x) = 2x$

Apply the quotient rule:

$du/dx = [8x (x^2 - 4) - 4x^2 2x] / (x^2 - 4)^2$

Simplify numerator:

- First term: $8x (x^2 - 4) = 8x^3 - 32x$
- Second term: $4x^2 2x = 8x^3$

So,

$du/dx = [8x^3 - 32x - 8x^3] / (x^2 - 4)^2$

Notice that $8x^3$ cancels out:

$du/dx = (-32x) / (x^2 - 4)^2$

Step 3: Apply Chain Rule to Find dY/dx

Recall:

$Y = u^5$

So,

$dY/dx = 5u^4 \, du/dx$

Substitute u and du/dx:

$dY/dx = 5 [(4x^2) / (x^2 - 4)]^4 [-32x / (x^2 - 4)^2]$

Now, express the entire derivative explicitly:

$dY/dx = 5 (4x^2)^4 / (x^2 - 4)^4 (-32x) / (x^2 - 4)^2$

Combine the denominators:

The denominator becomes $(x^2 - 4)^4 (x^2 - 4)^2 = (x^2 - 4)^{6}$

Simplify numerator:

$(4x^2)^4 = 4^4 x^{8} = 256 x^{8}$

Putting it all together:

$dY/dx = 5 \, 256 x^{8} (-32x) / (x^2 - 4)^6$

Calculate constants:

5 256 = 1280

1280 (-32) = -40960

So,

$dY/dx = -40960 x^{9} / (x^2 - 4)^6$

Step 4: Final Expression of the Derivative

The derivative of the function $Y = (4x^2 / (x^2 - 4))^5$ is:

$dY/dx = -40960 x^9 / (x^2 - 4)^6$

This expression provides the rate of change of the original function with respect to x.

Additional Insights and Tips

1. Importance of Chain and Quotient Rules

This problem exemplifies how the chain rule and quotient rule work together. The quotient rule is used to differentiate the inner function u, and the chain rule is applied to handle the outer power function.

2. Algebraic Simplification

Simplifying complex functions before differentiating can make calculations more manageable, though in some cases, it may be preferable to differentiate directly.

3. Domain Considerations

Note that the original function involves division by $(x^2 - 4)$, so the domain excludes $x = \pm 2$, where the denominator becomes zero. When analyzing the derivative, these points are critical as they correspond to vertical asymptotes or discontinuities.

Related Concepts and Further Reading

- **Product Rule:** Used when differentiating products of functions.
- **Chain Rule:** Essential for differentiating composite functions like the one discussed.
- **Quotient Rule:** Necessary when differentiating ratios of functions.
- **Implicit Differentiation:** Useful for functions defined implicitly.
- **Higher-Order Derivatives:** Derivatives of derivatives, important in analyzing function concavity and inflection points.

Practice Problems for Mastery

To solidify your understanding, consider practicing with similar functions:

1. Find the derivative of Y = (3x^3 / (x^2 + 1))^4
2. Differentiate Y = (sin x / x)^2
3. Compute the derivative of Y = (x^2 + 2x + 1)^3

Practicing these problems will help reinforce the application of chain rule, quotient rule, and algebraic manipulation skills.

Conclusion

Finding the derivative of complex functions like Y = (4x^2 / (x^2 - 4))^5 requires a systematic approach involving the quotient rule, chain rule, and algebraic simplification. By carefully identifying the inner functions and applying differentiation rules step-by-step, you can derive a precise expression for the rate of change. Mastery of these techniques is fundamental in calculus and essential for solving a broad range of problems involving rates, slopes, and tangents. Keep practicing with various functions to enhance your calculus proficiency and build confidence in handling derivatives of complex expressions.

Frequently Asked Questions

What is the main goal when finding the derivative of the function Y=(x^2 + 4)/(x^2 - 4)^5?

The main goal is to differentiate the given function with respect to x, applying the quotient rule and chain rule as needed to find its derivative.

What method should be used to differentiate Y=(x^2 + 4)/(x^2 - 4)^5?

The quotient rule should be used, along with the chain rule for the denominator raised to the fifth power.

How do you simplify the derivative of the function Y=(x^2 + 4)/(x^2 - 4)^5?

After applying the quotient rule, simplify the numerator and denominator, possibly factoring or combining like terms to present the derivative in a simplified form.

Does the function Y=(x^2 + 4)/(x^2 - 4)^5 have any points of non-differentiability?

Yes, points where the denominator equals zero, i.e., x=±2, are points where the function is not differentiable due to discontinuity.

What is the derivative of the numerator, x^2 + 4?

The derivative of the numerator x^2 + 4 is 2x.

How does the chain rule apply when differentiating (x^2 - 4)^5?

The chain rule states that the derivative of (x^2 - 4)^5 is 5(x^2 - 4)^4 times the derivative of x^2 - 4, which is 2x.

Can the derivative be expressed in a factored form?

Yes, after applying the quotient rule and simplifying, the derivative can often be expressed as a single fraction with factored numerator and denominator.

What is the significance of the power 5 in the function?

The power 5 indicates the function involves a fifth power in the denominator, which affects the derivative by introducing factors from the chain rule related to the fifth power.

What are the key steps to compute the derivative of Y=(x^2 + 4)/(x^2 - 4)^5?

The key steps are: 1) Apply the quotient rule, 2) Differentiate numerator and denominator separately using the chain rule, 3) Simplify the resulting expression.

Additional Resources

Find the Derivative of the Function \(Y = \left(\frac{x^2 + 4}{x^2 - 4} \right)^5 \)

When tackling the problem of differentiating complex functions like \(Y = \left(\frac{x^2 + 4}{x^2 - 4} \right)^5 \), understanding the underlying techniques such as the chain rule, quotient rule, and algebraic simplification becomes crucial. This type of differentiation not only exemplifies core calculus concepts but also sharpens problem-solving skills essential for higher-level mathematics and applications. In this comprehensive review, we'll explore the step-by-step process of finding the derivative of this function, analyze the mathematical strategies involved, and discuss the broader implications of mastering such derivatives.

Understanding the Function and Its Components

Before diving into the differentiation process, it’s vital to understand the structure of the function:

$$Y = \left(\frac{x^2 + 4}{x^2 - 4} \right)^5$$

This function is a composition of multiple functions:

- The inner rational function: $u(x) = \frac{x^2 + 4}{x^2 - 4}$
- The outer power function: $Y = [u(x)]^5$

Recognizing this nested structure suggests that the chain rule will be central to differentiation. Additionally, since $u(x)$ is a quotient of two polynomials, applying the quotient rule is necessary to find its derivative.

Step-by-Step Derivation Process

Step 1: Rewrite the Function for Clarity

Expressing the function explicitly:

$$Y = [u(x)]^5 \quad \text{where} \quad u(x) = \frac{x^2 + 4}{x^2 - 4}$$

This separation simplifies the differentiation process, allowing us to treat the outer and inner functions independently.

Step 2: Find the Derivative of the Outer Function $Y = [u(x)]^5$

Applying the chain rule, the derivative of Y with respect to x is:

$$\frac{dY}{dx} = 5 [u(x)]^4 \cdot \frac{du}{dx}$$

Thus, the key task reduces to finding $\frac{du}{dx}$.

Step 3: Differentiate the Inner Function $u(x)$ using the Quotient Rule

Recall the quotient rule:

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Let:

- $f(x) = x^2 + 4$, so $f'(x) = 2x$
- $g(x) = x^2 - 4$, so $g'(x) = 2x$

Applying the quotient rule:

$$\frac{du}{dx} = \frac{(2x)(x^2 - 4) - (x^2 + 4)(2x)}{(x^2 - 4)^2}$$

Factor $2x$ from numerator:

$$\frac{du}{dx} = \frac{2x[(x^2 - 4) - (x^2 + 4)]}{(x^2 - 4)^2}$$

Simplify inside the brackets:

$$(x^2 - 4) - (x^2 + 4) = x^2 - 4 - x^2 - 4 = -8$$

Therefore:

$$\frac{du}{dx} = \frac{2x \times (-8)}{(x^2 - 4)^2} = \frac{-16x}{(x^2 - 4)^2}$$

Assembling the Final Derivative Expression

Now, substitute \(\frac{du}{dx} \) back into the derivative of \(Y \):

\[
\frac{dY}{dx} = 5 [u(x)]^4 \times \left(\frac{-16x}{(x^2 - 4)^2} \right)
\]

Recall that \(u(x) = \frac{x^2 + 4}{x^2 - 4} \), so:

\[
\boxed{
\frac{dY}{dx} = -80x \left(\frac{x^2 + 4}{x^2 - 4} \right)^4 \times
\frac{1}{(x^2 - 4)^2}
}
\]

To write this more compactly, combine the denominators:

\[
\frac{dY}{dx} = -80x \left(\frac{x^2 + 4}{x^2 - 4} \right)^4 \times
\frac{1}{(x^2 - 4)^2}
\]

Expressed as a single fraction:

\[
\frac{dY}{dx} = -80x \times (x^2 + 4)^4 \bigg/ (x^2 - 4)^4
\]

since:

\[
\left(\frac{x^2 + 4}{x^2 - 4} \right)^4 = \frac{(x^2 + 4)^4}{(x^2 - 4)^4}
\]

and multiplying by \(1/(x^2 - 4)^2 \):

\[
\frac{(x^2 + 4)^4}{(x^2 - 4)^4} \times \frac{1}{(x^2 - 4)^2} = \frac{(x^2 +
4)^4}{(x^2 - 4)^6}
\]

But since the previous step was to keep it in a simplified form, the key is recognizing the overall power of the denominator.

Final Expression of the Derivative

The derivative of $Y = \left(\frac{x^2 + 4}{x^2 - 4} \right)^5$ is:

$$
\boxed{
\frac{dY}{dx} = -80x \times \frac{(x^2 + 4)^4}{(x^2 - 4)^4}
}
$$

This expression succinctly captures the rate of change of the function at any valid x, excluding points where the denominator becomes zero (i.e., $x = \pm 2$), which are points of discontinuity.

Mathematical Techniques Involved

The process of differentiating this complex function hinges on several fundamental calculus techniques:

Chain Rule

- Essential for differentiating composite functions.
- Applied when differentiating the outer power function $[u(x)]^5$.

Quotient Rule

- Used to find $\frac{du}{dx}$, where $u(x)$ is a quotient of polynomials.
- Involves differentiating numerator and denominator separately and combining.

Algebraic Simplification

- Simplifying the derivative expression into a more manageable form.
- Recognizing the common powers and factoring to write the derivative neatly.

Features, Pros, and Cons of This Differentiation Approach

Features:

- Combines multiple differentiation rules seamlessly.
- Demonstrates the power of algebraic manipulation in calculus.
- Provides a template for differentiating similar nested functions.

Pros:

- Clear step-by-step methodology enhances understanding.
- The final expression is compact and ready for further analysis.
- Highlights the importance of recognizing function structure.

Cons:

- The process can be algebraically intensive, especially for more complicated functions.
- Potential for mistakes in applying the quotient rule or algebraic simplification.
- Points of discontinuity (like $x = \pm 2$) need special attention, which may be overlooked.

Broader Implications and Applications

Mastering the differentiation of complex functions like this has significant implications:

- Mathematical Analysis: Essential for understanding the behavior of rational functions, including critical points, asymptotes, and monotonicity.
- Physics & Engineering: Rate of change calculations involving ratios of quantities.
- Economics & Biology: Modeling systems where variables are ratios or nested functions.
- Further Calculus Topics: Foundation for integration, optimization, and differential equations involving similar functions.

Conclusion

Differentiating $Y = \left(\frac{x^2 + 4}{x^2 - 4} \right)^5$

illustrates the elegance and power of calculus when multiple rules are combined effectively. By breaking down the problem into manageable parts—identifying the inner and outer functions, applying the quotient rule to the inner function, and then the chain rule to the outer function—we can derive an explicit formula for the derivative. This process reinforces key mathematical skills and deepens understanding of how complex functions behave and change. As with any advanced calculus problem, attention to detail and algebraic precision are paramount, but the resulting insights are invaluable for both

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