Find Two Integers Whose Sum Is 3 And Product Is -108.

Find Two Integers Whose Sum Is 3 And Product Is -108.
This problem is a classic example of solving a system of equations involving two unknown integers. Such problems often appear in algebra, and understanding how to approach them can enhance problem-solving skills significantly. In this article, we will explore the methods to find two integers that satisfy the conditions: their sum is 3, and their product is -108. We will delve into various techniques, including algebraic methods, factorization, and practical strategies to make solving similar problems easier in the future.

Understanding the Problem

Before jumping into solutions, it's essential to understand what the problem asks. We need to find two integers, say x and y, such that:

- The sum of these integers is 3:
$$x + y = 3$$
- The product of these integers is -108:
$$xy = -108$$

Our goal is to find all integer pairs (x, y) that satisfy both conditions simultaneously.

Approach to Solving the Problem

There are several methods to approach such problems. The most common and effective methods include:

- Using substitution
- Factoring quadratic equations
- Utilizing properties of integers and divisibility

Let's explore each method in detail.

Method 1: Using Algebraic Substitution

This method involves expressing one variable in terms of the other and then solving the resulting quadratic equation.

Step 1: Express one variable in terms of the other

From the sum equation:
$$x + y = 3 \implies y = 3 - x$$

Step 2: Substitute into the product equation

Plugging $y = 3 - x$ into the product equation:
$$xy = -108$$
$$x(3 - x) = -108$$

Step 3: Simplify to form a quadratic equation

$$3x - x^2 = -108$$
$$-x^2 + 3x + 108 = 0$$
Multiply through by -1 to make the quadratic standard:
$$x^2 - 3x - 108 = 0$$

Step 4: Solve the quadratic equation

Using the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
where $a=1$, $b=-3$, $c=-108$:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 1 \times (-108)}}{2 \times 1}$$
$$x = \frac{3 \pm \sqrt{9 + 432}}{2}$$
$$x = \frac{3 \pm \sqrt{441}}{2}$$
$$x = \frac{3 \pm 21}{2}$$

Calculate both solutions:

1. $x = \frac{3 + 21}{2} = \frac{24}{2} = 12$
2. $x = \frac{3 - 21}{2} = \frac{-18}{2} = -9$

Step 5: Find corresponding y values

Recall $y = 3 - x$:

- For $x = 12$:
$y = 3 - 12 = -9$

- For $x = -9$:
$y = 3 - (-9) = 12$

Result:

The pairs of integers are (12, -9) and (-9, 12).

Since the order of the integers doesn't matter for the product and sum, the solutions are:

- (12, -9)
- (-9, 12)

Summary of Method 1:
By substitution and quadratic formula, we found the integer pairs that satisfy the given sum and product.

Method 2: Factoring the Quadratic Equation

Another approach involves factoring the quadratic directly, especially since the quadratic has integer coefficients.

Step 1: Write the quadratic equation

$$x^2 - 3x - 108 = 0$$

Step 2: Find two numbers that multiply to -108 and add to -3

We need two integers m and n such that:

- $m \times n = -108$
- $m + n = -3$

Let's find all factors of -108 and check their sums.

Factors of -108:

108 factors: 1, 2, 3, 4, 6, 9, 12, 18, 27, 36, 54, 108

Considering negative factors as well:

- (-1, 108): sum = 107
- (1, -108): sum = -107
- (-2, 54): sum = 52
- (2, -54): sum = -52
- (-3, 36): sum = 33
- (3, -36): sum = -33
- (-4, 27): sum = 23
- (4, -27): sum = -23
- (-6, 18): sum = 12
- (6, -18): sum = -12
- (-9, 12): sum = 3
- (9, -12): sum = -3

Looking at the sums, the pair $(-9, 12)$ adds up to 3, which matches our required sum.

Step 3: Write the factorization

Since $m = -9$ and $n = 12$:

$$x^2 - 3x - 108 = (x - m)(x - n) = (x - (-9))(x - 12) = (x + 9)(x - 12)$$

Step 4: Find the roots

Set each factor to zero:

- $x + 9 = 0 \implies x = -9$
- $x - 12 = 0 \implies x = 12$

Corresponding y values:

- When $x = -9$, $y = 3 - x = 3 - (-9) = 12$
- When $x = 12$, $y = 3 - 12 = -9$

Again, the solutions are (12, -9) and (-9, 12).

Summary of Solutions

Both methods lead us to the same pair of solutions:

- (12, -9)
- (-9, 12)

These are the two integers whose sum is 3 and whose product is -108.

Verifying the Solutions

Always verify your solutions to ensure correctness.

For (12, -9):

- Sum: $12 + (-9) = 3$
- Product: $12 \times -9 = -108$

For (-9, 12):

- Sum: $-9 + 12 = 3$
- Product: $-9 \times 12 = -108$

Both pairs satisfy the original conditions.

Additional Tips for Solving Similar Problems

- Factorization is key: When dealing with quadratic equations, always look for factor pairs of the constant term that satisfy the sum condition.
- Use the quadratic formula when necessary: Especially when factorization isn't straightforward.
- Check for integer solutions: Not all solutions to quadratic equations are integers; verify solutions to ensure they meet the problem's requirements.
- Consider symmetry: Since the sum and product are symmetric in the variables, solutions often come in pairs like (x, y) and (y, x).

Conclusion

Finding two integers with a specific sum and product involves strategic algebraic manipulation and factorization techniques. In this case, the pairs $(12, -9)$ and $(-9, 12)$ satisfy the conditions that their sum is 3, and their product is -108. Mastering these methods enhances your ability to solve a wide range of algebraic problems involving systems of equations, quadratic equations, and integer solutions. Practice solving similar problems to build confidence and improve your problem-solving skills in algebra.

Frequently Asked Questions

How can I find two integers whose sum is 3 and product is -108?

Set up the equations: $x + y = 3$ and $xy = -108$. Then, find integers satisfying these conditions, such as solving the quadratic $x^2 - 3x - 108 = 0$.

What is the method to solve for two integers given their sum and product?

Use the relationships to form a quadratic equation: $x^2 - (\text{sum})x + (\text{product}) = 0$, then factor or use the quadratic formula to find the solutions.

Can you provide the specific integers whose sum is 3 and product is -108?

Yes, the integers are 12 and -9, since $12 + (-9) = 3$ and $12 (-9) = -108$.

Are the integers 12 and -9 the only solutions to this problem?

Yes, since the quadratic factors uniquely to these integers, they are the only integer solutions satisfying the conditions.

How do I verify that the integers 12 and -9 satisfy the conditions?

Add them: $12 + (-9) = 3$, and multiply: $12 (-9) = -108$. Both conditions are met, confirming the solution.

What is the quadratic equation derived from the given conditions?

The quadratic is $x^2 - 3x - 108 = 0$, obtained from the relationships $x + y = 3$ and $xy = -108$.

How can I solve the quadratic $x^2 - 3x - 108 = 0$?

Use the quadratic formula: $x = [3 \pm \sqrt{(9 + 432)}] / 2$, which simplifies to $x = [3 \pm \sqrt{441}] / 2$, leading to $x = [3 \pm 21] / 2$.

What are the solutions for x from the quadratic formula in this problem?

The solutions are $x = (3 + 21)/2 = 12$ and $x = (3 - 21)/2 = -9$, corresponding to the integer pairs (12, -9) and (-9, 12).

Are there any other pairs of integers that satisfy the sum and product conditions?

No, only the pair (12, -9) and its order counterpart satisfy both conditions as integers.

Additional Resources

Find Two Integers Whose Sum Is 3 And Product Is -108: An In-Depth Mathematical Exploration

In the realm of algebra, solving for unknowns often requires a delicate interplay between intuition and systematic analysis. Among the classic problems that epitomize this balance are those involving finding two numbers based on their sum and product. Specifically, the problem statement: "Find two integers whose sum is 3 and product is -108" offers a compelling case study, blending foundational algebraic principles with problem-solving strategies. This article embarks on a comprehensive journey to understand, analyze, and solve this problem, providing insights not only into the solution but also into the broader mathematical concepts it embodies.

Understanding the Problem: Clarifying the

Requirements

Before diving into the solution, it is crucial to interpret the problem statement accurately and understand what is being asked.

Defining the Variables

Let’s denote the two integers as x and y. The problem provides two conditions:
- Their sum: $x + y = 3$
- Their product: $xy = -108$

The goal is to find all integer pairs (x, y) satisfying both conditions simultaneously.

Why Focus on Integers?

The restriction that the numbers are integers narrows down the possible solutions significantly. Unlike real numbers, integers are discrete, which means the solution set is finite and can often be determined through logical deduction or systematic enumeration.

Mathematical Foundations: Formulating the Problem

Transforming the problem into algebraic equations sets the stage for analytical solutions.

Using Symmetric Equations

The two conditions can be expressed as:

$$
\begin{cases}
x + y = 3 \\
xy = -108
\end{cases}
$$

These are symmetric in x and y, which suggests the utility of quadratic equations. Recognizing this symmetry allows us to consider one variable in terms of the other or to formulate a quadratic equation with roots x and y.

Formulating a Quadratic Equation

Given the sum and product, we can construct a quadratic equation whose roots are x and y:

$$
t^2 - (x + y)t + xy = 0
$$

Substituting the known values:

$$
t^2 - 3t - 108 = 0
$$

The roots of this quadratic are precisely the integers x and y we seek, provided that these roots are integers themselves.

Solving the Quadratic Equation: The Key Step

The quadratic equation $t^2 - 3t - 108 = 0$ is central to finding the solutions.

Applying the Quadratic Formula

The quadratic formula:

$$
t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

For our equation $t^2 - 3t - 108 = 0$, the coefficients are:
- $a = 1$
- $b = -3$
- $c = -108$

Calculating the discriminant:

$$
\Delta = b^2 - 4ac = (-3)^2 - 4 \times 1 \times (-108) = 9 + 432 = 441
$$

Since 441 is a perfect square (21^2), the roots are rational and potentially integers.

Calculating the roots:

$$
t = \frac{3 \pm \sqrt{441}}{2} = \frac{3 \pm 21}{2}
$$

This yields two solutions:

$$
t_1 = \frac{3 + 21}{2} = \frac{24}{2} = 12
$$
$$
t_2 = \frac{3 - 21}{2} = \frac{-18}{2} = -9
$$

These roots suggest the pairings of x and y could be $(12, -9)$ or $(-9, 12)$.

Verifying the Solutions: Are They Valid?

Having identified potential solutions, the next step is to verify whether these pairs satisfy the original conditions, especially the integrality constraint.

Checking the Sum and Product

- For $(12, -9)$:
- Sum: $12 + (-9) = 3$
- Product: $12 \times -9 = -108$

- For $(-9, 12)$:
- Sum: $-9 + 12 = 3$
- Product: $-9 \times 12 = -108$

Both pairs satisfy the conditions and are integers, confirming their validity.

Are There Other Solutions?

Given the quadratic roots derived directly from the polynomial, these solutions are exhaustive. Since the quadratic's discriminant is a perfect square and the roots are integers, no other integer solutions exist for the pair (x, y).

Broader Implications and Related Concepts

This problem exemplifies several core algebraic concepts and problem-solving strategies that extend beyond this particular question.

Quadratic Equations and Roots

The process of translating a problem about sums and products into a quadratic equation demonstrates the power of algebraic transformations. Recognizing that the roots of quadratic equations relate directly to sum and product simplifies the process of solving for pairs of numbers.

Discriminant Analysis

The discriminant's role in determining the nature of roots (real, rational, integer, complex) is fundamental. In this case, the perfect square discriminant indicates the roots are rational and, more specifically, integers—making the solution straightforward.

Integer Constraints and Solution Uniqueness

Imposing the restriction that solutions are integers often limits the solution set dramatically. Here, the algebraic approach reveals that the solutions are unique and can be explicitly calculated.

Applications in Real-World Contexts

Problems of this type have applications in various fields such as engineering, physics, and computer science, where relationships between quantities are modeled through equations. For example, in balancing chemical equations, financial calculations, or designing algorithms that require integer solutions.

Additional Mathematical Explorations

While the primary approach yields the solutions, several related explorations can deepen understanding.

1. General Pattern Recognition

Given any sum S and product P, the quadratic:

$$
t^2 - St + P = 0
$$

can be analyzed similarly. The solutions' nature depends on the discriminant:

$$
\Delta = S^2 - 4P
$$

If \(\Delta\) is a perfect square, and roots are integers, then solutions exist accordingly.

2. Extending to Rational or Real Solutions

If the problem relaxes the integer constraint:
- Roots are rational if \(\sqrt{\Delta}\) is rational.
- Roots are real if \(\Delta \geq 0\).

This opens broader solution sets but is less restrictive.

3. Exploring Negative and Zero Values

In this problem, the negative product indicates one number is positive and the other negative, given the sum is positive. Exploring the signs of solutions offers insights into the structure of solutions.

Conclusion: The Elegance of Algebra in Problem Solving

The solution to the problem of finding two integers whose sum is 3 and product is -108 underscores the elegance and efficiency of algebraic methods. By translating the problem into a quadratic equation and analyzing its discriminant, we quickly identified the solutions without exhaustive enumeration. This approach exemplifies how fundamental algebraic concepts serve as powerful tools in problem-solving, enabling us to navigate from complex conditions to clear, definitive answers.

Moreover, the exploration highlights the importance of understanding the properties of quadratic equations, discriminants, and roots—skills that are vital across a myriad of mathematical and real-world applications. Whether in theoretical pursuits or practical problem-solving, mastering these techniques enhances analytical capabilities and fosters deeper appreciation for the interconnectedness of mathematical ideas.

In essence, the journey from problem statement to solution showcases the beauty of algebra: its capacity to distill complex relationships into manageable equations, guiding us to uncover hidden solutions with clarity and precision.

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