Prove That 3(x+1)(x+7)-(2x+5) ^2 Is Never Positive

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Understanding the behavior of algebraic expressions is fundamental in mathematics, especially in the fields of algebra and calculus. One intriguing question mathematicians often explore is whether certain expressions can take positive values, remain non-negative, or are always non-positive. In this article, we delve into the expression:

$$
3(x+1)(x+7) - (2x+5)^2
$$

Our goal is to rigorously prove that this expression is never positive for any real value of x. This involves analyzing the quadratic form, simplifying the expression, and understanding its graph to ascertain its maximum value and the nature of its positivity.

Understanding the Expression

Before diving into the proof, it’s essential to understand the components of the expression:

$$
E(x) = 3(x+1)(x+7) - (2x+5)^2
$$

- The first part, $3(x+1)(x+7)$, is a quadratic expression scaled by 3.
- The second part, $(2x+5)^2$, is a perfect square, which is always non-negative for all real x.

Since the squared term $(2x+5)^2$ is always ≥ 0, the behavior of $E(x)$ hinges on the interplay between these two parts. Our task is to show that for all real x, $E(x) \leq 0$, which means the expression is never positive.

Step 1: Expand and Simplify the Expression

To analyze $E(x)$, we first expand the terms:

$$
3(x+1)(x+7) = 3[x^2 + 7x + x + 7] = 3[x^2 + 8x + 7] = 3x^2 + 24x + 21
$$

Next, expand $(2x+5)^2$:

$$
(2x+5)^2 = 4x^2 + 20x + 25
$$

Now, write $E(x)$ as:

$$
E(x) = (3x^2 + 24x + 21) - (4x^2 + 20x + 25)
$$

Combine like terms:

$$
E(x) = 3x^2 + 24x + 21 - 4x^2 - 20x - 25
$$

Simplify:

$$
E(x) = (3x^2 - 4x^2) + (24x - 20x) + (21 - 25) = -x^2 + 4x - 4
$$

So, the expression simplifies to:

$$
E(x) = -x^2 + 4x - 4
$$

Step 2: Analyze the Quadratic Function

The simplified form:

$$
E(x) = -x^2 + 4x - 4
$$

is a quadratic function with the following characteristics:

- Leading coefficient $a = -1$, which is negative.
- This indicates that the parabola opens downward, meaning it has a maximum point.

To understand whether $E(x)$ can be positive, we need to find its maximum value and check if this maximum is greater than zero.

Quadratic Vertex Formula

The vertex of a quadratic $ax^2 + bx + c$ occurs at:

$$
x_{vertex} = -\frac{b}{2a}
$$

For our quadratic:

$$
a = -1, \quad b = 4, \quad c = -4
$$

Calculate x_{vertex}:

$$
x_{v} = -\frac{4}{2 \times (-1)} = -\frac{4}{-2} = 2
$$

Step 3: Find the Maximum Value of $E(x)$

Evaluate $E(x)$ at $x = 2$:

$$
E(2) = - (2)^2 + 4 \times 2 - 4 = -4 + 8 - 4 = 0
$$

The maximum value of $E(x)$ is 0, attained at $x=2$.

Step 4: Interpretation and Conclusion

Since the quadratic opens downward and its maximum value is 0 at \(x=2\), we conclude:

\[
\boxed{
\text{For all real } x, \quad E(x) \leq 0
}
\]

This directly implies that:

\[
3(x+1)(x+7) - (2x+5)^2 \leq 0
\]

and the expression is never positive.

Furthermore, the maximum value of 0 is achieved at \(x=2\), confirming that the expression equals zero at that point and is negative elsewhere.

Implications of the Result

This proof demonstrates a few key insights:

- The expression in question is a quadratic function with a negative leading coefficient, ensuring it opens downward.
- Its maximum point (vertex) provides the highest value it attains, which in this case is zero.
- Since the maximum is zero and the parabola opens downward, the function never exceeds zero, confirming it is never positive.

This type of analysis is fundamental in inequality proofs, optimization problems, and understanding the behavior of quadratic functions.

Additional Context and Applications

Understanding whether an algebraic expression can be positive is crucial in various mathematical and applied contexts, such as:

- Inequality proofs: Confirming the sign of expressions to establish inequalities.
- Optimization problems: Finding maximum or minimum values of functions.
- Physics and engineering: Analyzing stability conditions where certain parameters should not exceed specific bounds.
- Economics: Ensuring profit or cost functions do not surpass certain thresholds.

The methodology used here — simplifying the expression, analyzing the quadratic form, and finding its vertex — is a common technique in these areas.

Summary

- The original expression simplifies to $-x^2 + 4x - 4$.
- It is a downward-opening parabola with a maximum value at $x=2$.
- The maximum value of the expression is 0, achieved at $x=2$.
- Therefore, for all real x, the expression never takes positive values.

This rigorous approach confirms that:

$$
\boxed{
\text{Prove that } 3(x+1)(x+7) - (2x+5)^2 \text{ is never positive}
}
$$

is true, providing a comprehensive understanding of the quadratic behavior and its implications.

Final Remarks

Such proofs reinforce the importance of algebraic manipulation, understanding quadratic functions, and analyzing their graphs. They are fundamental skills for students and professionals working with inequalities, optimization, and mathematical modeling. Recognizing the properties of the functions involved enables mathematicians to draw meaningful conclusions about the behavior of complex expressions in various real-world applications.

Frequently Asked Questions

How can we prove that the expression 3(x+1)(x+7) - (2x+5)^2 is never positive?

By simplifying the expression and analyzing its quadratic form, we can show that it is always less than or equal to zero for all real x, thus never positive.

What is the first step in proving that 3(x+1)(x+7) - (2x+5)^2 ≤ 0 for all x?

The first step is to expand both parts of the expression to rewrite it as a quadratic in x.

How do we expand 3(x+1)(x+7) and (2x+5)^2?

We expand 3(x+1)(x+7) as 3[x^2 + 8x + 7] = 3x^2 + 24x + 21, and (2x+5)^2 as 4x^2 + 20x + 25.

After expansion, what does the expression become?

It becomes 3x^2 + 24x + 21 - (4x^2 + 20x + 25), which simplifies to -x^2 + 4x - 4.

How do we analyze the quadratic -x^2 + 4x - 4 to determine its sign?

We can complete the square or find its vertex to see whether it is always less than or equal to zero.

What is the vertex of the quadratic -x^2 + 4x - 4, and what does it tell us?

The vertex is at x = -b/(2a) = -4/(2 -1) = 2. Substituting x=2 gives the maximum value, which is 0, indicating the quadratic never exceeds zero.

Does the quadratic -x^2 + 4x - 4 ever become positive?

No, since its maximum value is 0 at x=2 and it opens downward, it is never positive but always less than or equal to zero.

What is the conclusion about the original expression

3(x+1)(x+7) - (2x+5)^2?

The expression is never positive for any real value of x; it is always less than or equal to zero.

Are there specific values of x where the expression equals zero?

Yes, at x=2, the expression equals zero, indicating the maximum point of the quadratic where the expression is not positive.

Additional Resources

Prove That 3(x+1)(x+7)-(2x+5)^2 Is Never Positive

Mathematics often challenges us with expressions that seem complicated at first glance but reveal interesting properties upon closer inspection. One such intriguing expression is 3(x+1)(x+7) - (2x+5)^2, and the task of proving that it is never positive is a classic example of how algebraic manipulation and understanding quadratic behavior can lead to elegant conclusions. In this comprehensive review, we will explore the structure of this expression, analyze its behavior across the real number line, and demonstrate rigorously why it cannot attain positive values. This investigation not only deepens our understanding of quadratic expressions but also exemplifies fundamental algebraic techniques applicable to a wide range of mathematical problems.

Understanding the Expression

Before delving into the proof, it's essential to understand the nature of the given expression:

$$ 3(x+1)(x+7) - (2x+5)^2 $$

This expression involves two quadratics: one arising from the product $(x+1)(x+7)$ scaled by 3, and the other being the square of a linear expression, $(2x+5)^2$. Recognizing the structure helps us identify the dominant terms and anticipate the overall behavior of the function.

Expanding and Simplifying

The first step in analyzing the expression is to expand both parts:

1. Expand $3(x+1)(x+7)$:

$$
(x+1)(x+7) = x^2 + 7x + x + 7 = x^2 + 8x + 7
$$
Multiplying by 3:

$$
3(x^2 + 8x + 7) = 3x^2 + 24x + 21
$$

2. Expand $(2x+5)^2$:

$$
(2x+5)^2 = 4x^2 + 20x + 25
$$

Putting it all together:

$$
3(x+1)(x+7) - (2x+5)^2 = (3x^2 + 24x + 21) - (4x^2 + 20x + 25)
$$

Subtracting term-by-term:

$$
(3x^2 - 4x^2) + (24x - 20x) + (21 - 25) = -x^2 + 4x - 4
$$

Thus, the original expression simplifies to:

$$
- x^2 + 4x - 4
$$

Analyzing the Simplified Quadratic

The problem reduces to analyzing the quadratic function:

$$
f(x) = -x^2 + 4x - 4
$$

Understanding the properties of this quadratic will allow us to determine whether the original expression can be positive.

Standard Form and Vertex

Expressing $f(x)$ in standard quadratic form:

$$
f(x) = -x^2 + 4x - 4
$$

This is a quadratic with a negative leading coefficient, indicating a parabola opening downward. Its maximum point (vertex) will give the highest value of the function, and if that maximum is non-positive, then the quadratic is never positive.

To find the vertex, use the vertex formula for $ax^2 + bx + c$:

$$
x_{v} = -\frac{b}{2a}
$$

Here,

$$
a = -1, \quad b = 4, \quad c = -4
$$

Calculating:

$$
x_{v} = -\frac{4}{2 \times (-1)} = -\frac{4}{-2} = 2
$$

Now, find the maximum value by substituting $x = 2$ into $f(x)$:

$$
f(2) = - (2)^2 + 4(2) - 4 = -4 + 8 - 4 = 0
$$

The vertex occurs at $(2, 0)$, and the maximum value of the quadratic is 0.

Implication of the Vertex Analysis

Since the maximum value of $f(x)$ is 0, and the parabola opens downward, it follows that:

$$
f(x) \leq 0 \quad \text{for all real } x

\]

In particular, \(f(x)\) is never positive; it is either zero at the vertex point \(x=2\) or negative elsewhere.

Conclusion: The Original Expression Is Never Positive

Given that the original expression simplifies exactly to \(f(x) = -x^2 + 4x - 4\), and we've shown that:

\[
f(x) \leq 0 \quad \text{for all } x \in \mathbb{R}
\]

it follows directly that:

\[
3(x+1)(x+7) - (2x+5)^2 \leq 0
\]

for all real numbers \(x\). Therefore, the expression is never positive. It reaches zero only at \(x=2\) and is negative for all other values.

Additional Insights and Features

Understanding the behavior of this quadratic offers valuable insights into similar algebraic problems. Here are some features and considerations:

- Quadratic Nature: Recognizing the form of the simplified expression as a quadratic is key to analyzing its maximum and minimum points.
- Vertex Analysis: The vertex provides the extremal value, which is crucial in determining whether the quadratic can be positive.
- Sign of Leading Coefficient: A negative leading coefficient indicates a downward-opening parabola, with a maximum at the vertex.
- Completing the Square: Rewriting the quadratic in completed square form can offer alternative insights, which we will explore below.

Completing the Square

Let's rewrite $f(x) = -x^2 + 4x - 4$ in completed square form:

$$
f(x) = - (x^2 - 4x + 4)
$$

Note that:

$$
x^2 - 4x + 4 = (x - 2)^2
$$

Thus,

$$
f(x) = - (x - 2)^2
$$

This form makes it transparent that $f(x) \leq 0$ with equality only when $(x - 2)^2 = 0 \Rightarrow x=2$. It confirms our earlier conclusion about the maximum point and the non-positivity of the expression.

Pros and Cons of the Approach

Pros:

- Clarity: The algebraic simplification and completing the square provide clear, direct evidence that the expression is never positive.
- Efficiency: The approach quickly reduces the problem to analyzing a single quadratic function.
- Generalizability: The technique can be applied to similar problems involving quadratic expressions.

Cons:

- Initial Assumption: The approach assumes familiarity with quadratic formulas and completing the square techniques.
- Limited Scope: The proof relies on algebraic manipulation; alternative methods might involve calculus or graphing for visual confirmation.

Alternative Methods

While algebraic manipulation is straightforward here, other methods can reinforce our conclusion:

- Graphical Analysis: Plotting the quadratic reveals the maximum point at zero, illustrating the non-positivity visually.
- Calculus-Based Proof: Deriving the derivative confirms the maximum at $x=2$, with the maximum value being zero.
- Inequality Approach: Using inequalities to bound the quadratic can also demonstrate it is never positive.

Final Remarks

The problem of proving that $3(x+1)(x+7) - (2x+5)^2$ is never positive is a compelling example of how algebraic techniques such as expansion, simplification, and completing the square can lead to elegant solutions. The key insight resides in recognizing the expression as a quadratic function with a maximum value of zero, achieved at a specific point. This confirms that the original expression is always less than or equal to zero, never crossing into positive territory.

This analysis not only resolves the specific problem but also illustrates a fundamental approach in algebra: transforming complex expressions into manageable forms to reveal their inherent properties. Such techniques are invaluable tools for students and mathematicians alike, fostering a deeper appreciation of the structure and behavior of polynomial functions.

In summary:

- The original expression simplifies to $-x^2 + 4x - 4$.
- The maximum value of this quadratic is zero, occurring at $x=2$.
- The quadratic is always less than or equal to zero, hence the original expression is never positive.
- The problem elegantly demonstrates the power of algebraic manipulation and quadratic analysis in mathematical proofs.

This comprehensive exploration underscores the importance of algebraic insight in establishing the non-positivity of quadratic expressions, reinforcing core concepts that underpin much of higher mathematics.

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