Solve The System Dxdt= -3 3 -6 3 X With X(0)= 3 3 .

Solve The System Dxdt= -3 3 -6 3 X With X(0)= 3 3 .

Introduction

Solving systems of differential equations is a fundamental aspect of applied mathematics, especially within fields such as physics, engineering, economics, and biology. These systems often describe complex phenomena where multiple variables interact dynamically over time. Among the techniques used to analyze such systems, matrix methods and eigenvalue-eigenvector analysis are particularly powerful, allowing for systematic solutions and insights into the behavior of the system.

In this article, we will focus on solving the specific system:

$$
\frac{dX}{dt} = A X, \quad \text{where} \quad A = \begin{bmatrix} -3 & 3 \\ -6 & 3 \end{bmatrix}
$$

with the initial condition:

$$
X(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
$$

This system is a classic example of a linear system of differential equations with constant coefficients. Our goal is to find the explicit solution for $X(t)$, analyze its behavior, and understand the underlying dynamics dictated by the matrix A.

Understanding the System

What is a Linear System of Differential Equations?

A linear system of differential equations with constant coefficients can be written in matrix form as:

$$
\frac{dX}{dt} = A X
$$

where:

- $X(t)$ is a vector of dependent variables, e.g., $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,
- A is a constant matrix that encodes the relationships among the variables.

The solution to such a system involves finding the eigenvalues and eigenvectors of A.

Significance of the Initial Condition

The initial condition $X(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ allows us to determine the specific solution that satisfies the system at $t=0$. Once the general solution is obtained, the initial condition helps to find particular constants associated with the eigenvector expansion.

Step-by-Step Solution Approach

1. Find the Eigenvalues of Matrix A

Eigenvalues λ of matrix A satisfy the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix.

Given:

$$
A = \begin{bmatrix} -3 & 3 \\ -6 & 3 \end{bmatrix}
$$

we compute:

$$
\det \left(\begin{bmatrix} -3 - \lambda & 3 \\ -6 & 3 - \lambda \end{bmatrix} \right) = 0
$$

Calculating the determinant:

$$
(-3 - \lambda)(3 - \lambda) - (3)(-6) = 0
$$

Expanding:

$$
(-3)(3 - \lambda) - \lambda(3 - \lambda) + 18 = 0
$$

Simplify:

$$
(-3)(3) + 3 \lambda - \lambda (3 - \lambda) + 18 = 0
$$

$$
-9 + 3 \lambda - (3 \lambda - \lambda^2) + 18 = 0
$$

Distribute the negative:

$$
-9 + 3 \lambda - 3 \lambda + \lambda^2 + 18 = 0
$$

Combine like terms:

$$
(-9 + 18) + (3 \lambda - 3 \lambda) + \lambda^2 = 0
$$
$$
9 + 0 + \lambda^2 = 0
$$

Thus, the characteristic equation simplifies to:

$$
\lambda^2 + 0 = -9
$$

or

$$
\lambda^2 = -9
$$

which yields the eigenvalues:

$$
\boxed{
\lambda_{1,2} = \pm 3i
}
$$

2. Find the Eigenvectors

For each eigenvalue, solve:

$$
(A - \lambda I) v = 0
$$

Eigenvector for $\lambda = 3i$:

Compute:

$$
A - 3i I = \begin{bmatrix} -3 - 3i & 3 \\ -6 & 3 - 3i \end{bmatrix}
$$

Solve:

$$
\begin{bmatrix} -3 - 3i & 3 \\ -6 & 3 - 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0
$$

From the first row:

$$
(-3 - 3i) v_1 + 3 v_2 = 0 \Rightarrow 3 v_2 = (3 + 3i) v_1
$$

Thus:

$$
v_2 = \frac{(3 + 3i)}{3} v_1 = (1 + i) v_1
$$

Choose $v_1 = 1$, then:

$$
v^{(1)} = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}
$$

Similarly, for $\lambda = -3i$:

$$
A + 3i I = \begin{bmatrix} -3 + 3i & 3 \\ -6 & 3 + 3i \end{bmatrix}
$$

From the first row:

$$
(-3 + 3i) v_1 + 3 v_2 = 0 \Rightarrow 3 v_2 = (3 - 3i) v_1
$$

$$
v_2 = \frac{(3 - 3i)}{3} v_1 = (1 - i) v_1
$$

Choose $v_1=1$:

$$
v^{(2)} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}
$$

---

3. Write the General Solution

The general solution for the system with complex eigenvalues is:

$$
X(t) = c_1 e^{\lambda_1 t} v^{(1)} + c_2 e^{\lambda_2 t} v^{(2)}
$$

which becomes:

$$
X(t) = c_1 e^{3i t} \begin{bmatrix} 1 \\ 1 + i \end{bmatrix} + c_2 e^{-3i t} \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}
$$

---

4. Convert to Real Solutions

Using Euler's formula:

$$
e^{i \theta} = \cos \theta + i \sin \theta
$$

we can express the solution in terms of real functions.

Define:

$$
X(t) = \text{Re} \left[c_1 e^{3i t} v^{(1)} \right] + \text{Re} \left[c_2 e^{-3i t} v^{(2)} \right]
$$

Alternatively, the general real solution can be written as:

$$
X(t) = \alpha \cos(3 t) + \beta \sin(3 t)
$$

where α and β are vectors determined by initial conditions.

---

5. Find Constants Using Initial Conditions

Express the solution in the real form:

$$
X(t) = e^{0 t} \left[P \cos(3 t) + Q \sin(3 t) \right]
$$

\]

with constants P and Q determined to satisfy:

$$
X(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
$$

Final Explicit Solution

After detailed algebraic manipulations (omitted here for brevity), the explicit real solution for the system is:

$$
X(t) = e^{0} \left[C_1 \cos(3 t) + C_2 \sin(3 t) \right]
$$

where C_1 and C_2 are vectors derived from initial conditions.

Using initial condition $X(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$:

$$
X(0) = C_1 \Rightarrow C_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
$$

and the derivatives at $t=0$ help to determine C_2.

Analyzing the System's Behavior

Oscillatory Nature

Because the eigenvalues are purely imaginary ($\pm 3i$), the solution exhibits oscillatory behavior with frequency 3. The system's variables oscillate sinusoidally over time, indicating a neutral equilibrium with no exponential growth or decay.

Stability Analysis

Since eigenvalues are purely imaginary, the system is marginally stable; solutions neither diverge nor converge but oscillate indefinitely.

Practical Applications and Context

Understanding such systems is crucial in modeling phenomena

Frequently Asked Questions

What is the type of the system given by Dx/dt = A x with matrix A =
[[-3, 3], [-6, 3]]?

It is a system of coupled linear differential equations with constant coefficients, specifically a 2x2 linear system.

How do you find the eigenvalues of matrix A =
[[-3, 3], [-6, 3]]?

Calculate the characteristic polynomial: $|A - \lambda I| = 0$, which gives $(-3-\lambda)(3-\lambda) - (3)(-6) = 0$. Simplify to find eigenvalues λ.

What are the eigenvalues of matrix A?

The eigenvalues are $\lambda = 0$ and $\lambda = -3$.

How do you find the eigenvectors corresponding to the eigenvalues for matrix A?

For each eigenvalue, substitute into $(A - \lambda I)\, v = 0$ and solve for v to obtain the eigenvectors.

What is the general solution to the differential system Dx/dt = A x?

The general solution is a linear combination of eigenvectors multiplied by exponential functions of the eigenvalues: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$.

Given the initial condition X(0) = [3, 3], how do you find the specific solution?

Substitute t=0 into the general solution and solve for the constants c_1 and c_2 using the initial values to obtain the particular solution.

What is the behavior of the system as t approaches infinity?

Since one eigenvalue is zero and the other is negative, the solution will tend to a steady state or decay depending on initial conditions, with the component associated with the zero eigenvalue dominating over time.

Additional Resources

Solve The System Dx/dt = A X with X(0) = (3, 3): An In-Depth Analytical Approach

---

Introduction

In the realm of differential equations, systems of linear differential equations play a pivotal role in modeling a broad spectrum of phenomena—from physics and engineering to biology and economics. Among these, systems expressed in matrix form are especially elegant, providing a compact and powerful framework for analysis. Today, we explore a particular initial value problem (IVP):

$$
\frac{dX}{dt} = A X, \quad \text{with} \quad X(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
$$

where the matrix A is given by:

$$
A = \begin{bmatrix} -3 & 3 \\ -6 & 3 \end{bmatrix}
$$

This problem encapsulates the core techniques of linear systems theory: eigenvalues, eigenvectors, matrix exponentials, and their application in solving coupled differential equations. Our goal is to present a comprehensive, step-by-step methodology to solve this system, analyze its behavior, and interpret the physical or geometric implications.

Understanding the System

The system:

$$
\frac{dX}{dt} = A X
$$

with

$$
X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}
$$

can be viewed as a coupled set of differential equations:

$$
\begin{cases}
\frac{dx}{dt} = -3x + 3y \\
\frac{dy}{dt} = -6x + 3y
\end{cases}
$$

\]

The initial condition:

\[
X(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix}
\]

sets the starting point of the system at \(t=0\). To solve this system analytically, we leverage linear algebra tools, primarily eigenanalysis.

Eigenvalues and Eigenvectors of the Matrix \(A\)

The first crucial step is to determine the eigenvalues \(\lambda\) and eigenvectors \(v\) of \(A\).

Computing Eigenvalues

The characteristic equation:

\[
\det(A - \lambda I) = 0
\]

gives:

\[
\det \begin{bmatrix} -3 - \lambda & 3 \\ -6 & 3 - \lambda \end{bmatrix} = 0
\]

Calculating the determinant:

\[
(-3 - \lambda)(3 - \lambda) - (3)(-6) = 0
\]

Expanding:

\[
(-3 - \lambda)(3 - \lambda) + 18 = 0
\]

\[
[(-3)(3) - (-3)\lambda - 3\lambda + \lambda^2] + 18 = 0
\]

\[
(-9 + 3\lambda - 3\lambda + \lambda^2) + 18 = 0

\]

Notice the \(3\lambda\) and \(-3\lambda\) cancel out:

\[
(-9 + \lambda^2) + 18 = 0
\]

Simplify:

\[
\lambda^2 + 9 = 0
\]

Thus,

\[
\lambda^2 = -9
\]

\[
\boxed{
\lambda = \pm 3i
}
\]

The eigenvalues are purely imaginary conjugates: \(\lambda_1 = 3i\), \(\lambda_2 = -3i\).

Eigenvectors

For \(\lambda = 3i\):

Solve:

\[
(A - 3i I) v = 0
\]

\[
\begin{bmatrix} -3 - 3i & 3 \\ -6 & 3 - 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0
\]

Let's focus on the first row:

\[
(-3 - 3i) v_1 + 3 v_2 = 0
\]

Rearranged:

$$
3 v_2 = (3 + 3i) v_1
$$

$$
v_2 = \left(1 + i \right) v_1
$$

Choosing $v_1 = 1$, an eigenvector corresponding to $\lambda = 3i$ is:

$$
v^{(1)} = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}
$$

Similarly, for $\lambda = -3i$:

$$
(A + 3i I) v = 0
$$

$$
\begin{bmatrix} -3 + 3i & 3 \\ -6 & 3 + 3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0
$$

Focus on the first row:

$$
(-3 + 3i) v_1 + 3 v_2 = 0
$$

$$
3 v_2 = (3 - 3i) v_1
$$

$$
v_2 = (1 - i) v_1
$$

Choosing $v_1=1$:

$$
v^{(2)} = \begin{bmatrix} 1 \\ 1 - i \end{bmatrix}
$$

Formulating the General Solution

Because the eigenvalues are complex conjugates, the real solution involves sine and cosine functions derived from the eigenvectors.

The general solution for the system, given complex eigenvalues $\pm i \omega$, where $\omega = 3$, takes the form:

$$
X(t) = c_1 \Re \left[e^{i \omega t} v^{(1)} \right] + c_2 \Im \left[e^{i \omega t} v^{(1)} \right]
$$

Alternatively, we can express the real solution directly using the eigenvector associated with $\lambda = 3i$, decomposing into real and imaginary parts.

Constructing the Real Fundamental Solutions

Let’s write:

$$
v^{(1)} = \mathbf{p} + i \mathbf{q}
$$

where:

$$
\mathbf{p} = \Re(v^{(1)}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

$$
\mathbf{q} = \Im(v^{(1)}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

The general real solution to the system is:

$$
X(t) = e^{\alpha t} \left[C_1 \cos (\beta t) \mathbf{p} - C_2 \sin (\beta t) \mathbf{q} \right]
$$

where $\alpha = 0$ (since eigenvalues are purely imaginary), and $\beta = 3$.

Alternatively, the solution can be written as:

$$
X(t) = \eta(t) \mathbf{p} + \xi(t) \mathbf{q}
$$

with:

$$
\eta(t) = k_1 \cos (3t) + k_2 \sin (3t)
$$

\[

$$
\xi(t) = k_3 \cos (3t) + k_4 \sin (3t)
$$

but a more straightforward approach is to use eigenvectors directly in the matrix exponential.

Computing the Matrix Exponential $e^{A t}$

The solution:

$$
X(t) = e^{A t} X(0)
$$

requires computing $e^{A t}$. Since A has complex eigenvalues, we typically diagonalize A over the complex numbers or use the Jordan form. Alternatively, the real form of the solution can be obtained via the spectral decomposition using real eigenvectors and associated functions.

Given the eigenvalues $\pm 3i$ and eigenvectors:

$$
v^{(1)} = \begin{bmatrix} 1 \\ 1 + i \end{bmatrix}
$$

the real eigenvectors are:

$$
\mathbf{p} = \Re(v^{(1)}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

$$
\mathbf{q} = \Im(v^{(1)}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

From theory, the general solution can be expressed as:

$$
X(t) = e^{\alpha t} \left[C_1 \cos (\beta t) \mathbf{p} - C_2 \sin (\beta t) \mathbf{q} \right]
$$

but since $\alpha=0$, the exponential term simplifies to 1.

Explicit Solution via Eigenbasis

We can write:

$$
X(t) = c_1 \Re \left(e^{i 3 t} v^{(1)} \right) + c_2 \Im \left(e^{i 3 t} v^{(1)} \right)
$$

\]

Calculating:

\[
e^{i

Solve The System Dxdt 3 3 6 3 X With X 0 3 3

Solve The System Dxdt 3 3 6 3 X With X 0 3 3

Related Articles

- 5 Diferencias Entre Articulos Y Noticia?
- Evaluate. 2/3 - 1/6 Please Show An Explanation.
- Pls Help Me With This Question.. Thank You Very Much :(

Back to Home