Calculate L4 For F(x)=6cos(x/2) Over [2/4,2/2][2/4,2/2].

Calculate L4 For F(x)=6cos(x/2) Over [2/4,2/2][2/4,2/2].

Introduction: Understanding the Problem and Its Significance

Calculating the fourth-degree Taylor polynomial (L4) for a function like $F(x) = 6 \cos(x/2)$ over a specified interval is a foundational task in approximation theory and numerical analysis. Such approximations are vital in computational mathematics, engineering, and physics to estimate function values with high efficiency and accuracy when exact calculations are complex or computationally expensive.

The focus here is to determine the L4 polynomial centered at a point within the interval—most commonly at the midpoint or a strategic point—then analyze its accuracy over the given interval $[2/4, 2/2]$. The interval, simplified, is $[0.5, 1]$.

This guide will walk through the step-by-step process of calculating the fourth-degree Taylor polynomial, including derivatives, the Taylor series expansion, and the error estimation, ensuring a comprehensive understanding for students and professionals alike.

Understanding the Function $F(x) = 6 \cos(x/2)$

Before diving into the Taylor polynomial, it’s essential to comprehend the properties of the function:

- Type of function: Trigonometric cosine function scaled by 6.
- Derivative pattern: Derivatives involve alternating sine and cosine functions with scaled arguments.
- Periodicity and symmetry: The cosine function exhibits periodic behavior, which influences the approximation accuracy over intervals.

Step 1: Choosing the Center Point for Taylor Expansion

The Taylor polynomial is typically centered at a point a within the interval $[0.5, 1]$. Common choices include:

- Midpoint: $a = 0.75$
- Endpoint: $a = 0.5$ or $a = 1$

For this example, we select $a = 0.75$, the midpoint, to minimize maximum approximation error across the interval.

Step 2: Calculating Derivatives of $F(x)$

To construct a 4th-degree Taylor polynomial, derivatives up to the 4th order are needed:

First derivative $F'(x)$:

$$
F'(x) = 6 \times \frac{d}{dx} \cos\left(\frac{x}{2}\right) = 6 \times \left(- \sin\left(\frac{x}{2}\right) \times \frac{1}{2} \right) = -3 \sin\left(\frac{x}{2}\right)
$$

Second derivative $F''(x)$:

$$
F''(x) = -3 \times \frac{d}{dx} \sin\left(\frac{x}{2}\right) = -3 \times \left(\cos\left(\frac{x}{2}\right) \times \frac{1}{2} \right) = -\frac{3}{2} \cos\left(\frac{x}{2}\right)
$$

Third derivative $F'''(x)$:

$$
F'''(x) = -\frac{3}{2} \times \frac{d}{dx} \cos\left(\frac{x}{2}\right) = -\frac{3}{2} \times \left(- \sin\left(\frac{x}{2}\right) \times \frac{1}{2} \right) = \frac{3}{4} \sin\left(\frac{x}{2}\right)
$$

Fourth derivative $F^{(4)}(x)$:

$$
F^{(4)}(x) = \frac{3}{4} \times \frac{d}{dx} \sin\left(\frac{x}{2}\right) = \frac{3}{4} \times \left(\cos\left(\frac{x}{2}\right) \times \frac{1}{2} \right) = \frac{3}{8} \cos\left(\frac{x}{2}\right)
$$

Step 3: Evaluating Derivatives at $a = 0.75$

Determine the derivatives at the center point:

$$
x = 0.75
$$

Calculate $\frac{x}{2} = \frac{0.75}{2} = 0.375$.

Evaluate the derivatives:

- $F(a) = 6 \cos(0.375)$
- $F'(a) = -3 \sin(0.375)$
- $F''(a) = -\frac{3}{2} \cos(0.375)$
- $F'''(a) = \frac{3}{4} \sin(0.375)$
- $F^{(4)}(a) = \frac{3}{8} \cos(0.375)$

Using approximate values:

- $\cos(0.375) \approx 0.9305$
- $\sin(0.375) \approx 0.3660$

Thus:

- $F(0.75) \approx 6 \times 0.9305 = 5.583$
- $F'(0.75) \approx -3 \times 0.3660 = -1.098$
- $F''(0.75) \approx -\frac{3}{2} \times 0.9305 = -1.39575$
- $F'''(0.75) \approx \frac{3}{4} \times 0.3660 = 0.2745$
- $F^{(4)}(0.75) \approx \frac{3}{8} \times 0.9305 = 0.348$

Step 4: Constructing the 4th-Degree Taylor Polynomial $L_4(x)$

The Taylor polynomial centered at $a = 0.75$:

$$
L_4(x) = F(a) + F'(a)(x - a) + \frac{F''(a)}{2!}(x - a)^2 + \frac{F'''(a)}{3!}(x - a)^3 + \frac{F^{(4)}(a)}{4!}(x - a)^4
$$

Plugging in the derivatives:

$$
L_4(x) = 5.583 - 1.098(x - 0.75) - \frac{1.39575}{2}(x - 0.75)^2 + \frac{0.2745}{6}(x - 0.75)^3 + \frac{0.348}{24}(x - 0.75)^4
$$

Simplify coefficients:

- $\frac{F''(a)}{2} = -0.6979$
- $\frac{F'''(a)}{6} = 0.04575$
- $\frac{F^{(4)}(a)}{24} = 0.0145$

Thus,

$$
L_4(x) \approx 5.583 - 1.098(x - 0.75) - 0.6979 (x - 0.75)^2 + 0.04575 (x - 0.75)^3 + 0.0145 (x - 0.75)^4
$$

Step 5: Approximate Function Values Using $L_4(x)$

To estimate $F(x)$ over the interval, compute $L_4(x)$ at various points $x \in [0.5, 1]$.

Example: $x = 0.5$

Calculate $x - 0.75 = -0.25$:

$$
L_4(0.5) \approx 5.583 - 1.098(-0.25) - 0.6979(0.0625) + 0.04575(-0.015625) + 0.0145(0.00390625)
$$

Compute step-by-step:

- $-1.098 \times -0.25 = 0.2745$
- $-0.6979 \times 0.0625 = -0.0436$

- $0.04575 \times -0.015625 = -0.000715$
- $0.0145 \times 0.00390625 = 0.000057$

Sum:

$$
L_4(0.5) \approx 5.583 + 0.2745 - 0.0436 - 0.000715 + 0.000057 \approx 5.583 + 0.2745 - 0.0436 - 0.000715 + 0.000057 \approx 5.814
$$

Compare with the actual function value:

$$
F(0.5) = 6 \cos(0.25) \approx 6 \times 0.9689 = 5.813
$$

The approximation is remarkably close.

Step 6: Error Estimation and Remainder Term

The accuracy of the Taylor polynomial depends on the remainder \(R_4(x

Frequently Asked Questions

What does L4 represent in the context of the function F(x)=6cos(x/2)?

L4 typically refers to the 4th degree Taylor polynomial (or Taylor series approximation) of the function F(x) centered at a specific point, used to approximate the function near that point.

How do I determine the Taylor polynomial of degree 4 for F(x)=6cos(x/2) over the interval [1/2, 1] (which is equivalent to [2/4, 2/2])?

To find the degree 4 Taylor polynomial (L4) of F(x)=6cos(x/2) over [1/2, 1], identify the center point (often x=0 or the midpoint of the interval), compute derivatives of F(x) up to the 4th order at that point, and construct the polynomial accordingly.

What is the significance of the interval [2/4, 2/2] in calculating L4 for F(x)=6cos(x/2)?

The interval [2/4, 2/2], equivalent to [0.5, 1], indicates the domain over which you want to approximate F(x) using the Taylor polynomial, ensuring the approximation is accurate within this range.

How do derivatives of F(x)=6cos(x/2) affect the

Taylor polynomial L4?

The derivatives of F(x)=6cos(x/2) determine the coefficients of the Taylor polynomial. For degree 4, you need to compute derivatives up to the 4th order at the center point, then use these to build the polynomial approximation.

Can you provide the steps to calculate L4 for F(x)=6cos(x/2) over the interval [0.5, 1]?

Yes. First, choose a center point (e.g., x=0.75). Next, compute F(x) and its derivatives up to 4th order at that point. Then, apply the Taylor polynomial formula to construct L4: sum of derivatives divided by factorial terms, multiplied by (x - center)^n.

What are the practical applications of calculating L4 for this function over the given interval?

Calculating L4 allows for an accurate approximation of F(x) within the interval, which is useful in numerical analysis, engineering, and physics when evaluating the function efficiently without complex calculations.

Additional Resources

Calculate L4 For F(x)=6cos(x/2) Over [2/4, 2/2]

Introduction

In the realm of numerical analysis, the precise approximation of functions over specific intervals is fundamental to various scientific, engineering, and mathematical applications. Among the array of techniques, polynomial interpolation—especially using Newton's or Lagrange's methods—stands out as a vital tool for estimating functions where exact calculations are impractical or unnecessary. A particularly insightful aspect of polynomial interpolation involves calculating the Lagrange polynomial of degree 4 (L4), which provides a cubic or quartic polynomial approximation of a function based on selected data points.

This article delves into the detailed process of calculating the L4 (Lagrange polynomial of degree 4) approximation for the function:

$$ F(x) = 6 \cos\left(\frac{x}{2}\right) $$

over the interval $\left[\frac{2}{4}, \frac{2}{2}\right]$, i.e., from $x=0.5$ to $x=1$. The goal is to explore the theoretical underpinnings, practical computation steps, and implications of this approximation, providing clarity for scholars and practitioners engaged in numerical analysis, approximation theory, and computational mathematics.

Setting the Stage: Understanding the Problem

The Function and Interval

The function under investigation is:

$$F(x) = 6 \cos\left(\frac{x}{2}\right)$$

defined over the interval:

$$[0.5, 1.0]$$

This interval is relatively narrow, yet it presents enough variability to warrant approximation techniques, especially since the cosine function exhibits oscillatory behavior that can be challenging to approximate precisely.

Objective

Calculate the Lagrange polynomial of degree 4 (L4) that interpolates $F(x)$ at five carefully chosen points within $[0.5, 1.0]$. The polynomial aims to closely approximate $F(x)$ across the entire interval.

Theoretical Foundations

Polynomial Interpolation and Lagrange Form

Given $n+1$ data points $\{(x_i, y_i)\}$, the Lagrange interpolation polynomial $L_n(x)$ of degree at most n is:

$$
L_n(x) = \sum_{i=0}^{n} y_i \cdot \ell_i(x)
$$

where each basis polynomial $\ell_i(x)$ is:

$$
\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}
$$

This form guarantees that $L_n(x_i) = y_i$ for each data point.

Choosing Data Points

To construct a degree 4 polynomial (L4), five data points are necessary:

$$
\{(x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}
$$

The points are typically selected to be equally spaced or strategically chosen to optimize approximation accuracy. For the interval $[0.5, 1.0]$, a natural choice is to select five points equidistantly:

Index	x_i	Calculation
0	$x_0 = 0.5$	Starting point
1	$x_1 = 0.625$	$0.5 + \frac{1}{4} \times 0.125$
2	$x_2 = 0.75$	$0.5 + 0.25 \times 0.5$
3	$x_3 = 0.875$	$0.5 + 0.75 \times 0.125$

| 4 | $x_4 = 1.0$ | End point |

Corresponding $y_i = F(x_i)$:

$$
y_i = 6 \cos\left(\frac{x_i}{2}\right)
$$

which can be computed explicitly.

Practical Calculation Process

Step 1: Compute Data Points

Let's evaluate $F(x)$ at the five points:

1. $x_0=0.5$:

$$
y_0 = 6 \cos\left(\frac{0.5}{2}\right) = 6 \cos(0.25) \approx 6 \times 0.968912 \approx 5.8135
$$

2. $x_1=0.625$:

$$
y_1 = 6 \cos(0.3125) \approx 6 \times 0.951056 \approx 5.7063
$$

3. $x_2=0.75$:

$$
y_2 = 6 \cos(0.375) \approx 6 \times 0.930518 \approx 5.5831
$$

4. $x_3=0.875$:

$$
y_3 = 6 \cos(0.4375) \approx 6 \times 0.905580 \approx 5.4335
$$

5. $x_4=1.0$:

$$
y_4 = 6 \cos(0.5) \approx 6 \times 0.877583 \approx 5.2655
$$

Step 2: Construct the Lagrange Basis Polynomials $\ell_i(x)$

For each i, compute:

$$
\ell_i(x) = \prod_{j=0, j \neq i}^{4} \frac{x - x_j}{x_i - x_j}
$$

This involves calculating the denominators:

$$
D_i = \prod_{j=0, j \neq i}^{4} (x_i - x_j)
$$

and then forming the basis polynomial.

Step 3: Explicitly Form the Interpolating Polynomial

The Lagrange polynomial is:

$$
L_4(x) = \sum_{i=0}^{4} y_i \cdot \ell_i(x)
$$

Given the explicit values, the polynomial can be expressed in the form:

$$
L_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4
$$

or retained in the Lagrange form for evaluation purposes.

Deep Dive: Numerical Computation and Error Analysis

Computing the Coefficients

The polynomial can be computed directly via symbolic algebra or numerical methods, such as polynomial fitting routines available in computational tools like MATLAB, Python's SciPy, or Mathematica. For illustration, a simplified approach involves the following:

- Calculate all basis polynomials $\ell_i(x)$.
- Multiply each basis polynomial by the corresponding y_i.
- Sum all to obtain $L_4(x)$.

This process ensures an exact interpolation at the five points.

Error Estimation and Bounds

The interpolation error $E(x)$ for a degree-4 polynomial is given by:

$$
E(x) = \frac{f^{(5)}(\xi)}{5!} \prod_{i=0}^4 (x - x_i)
$$

for some ξ in $[0.5, 1.0]$. Since $F(x)$ involves cosine, its fifth derivative can be computed and bounded to estimate the maximum error across the interval.

Implications and Applications

Approximation Accuracy

The degree-4 Lagrange polynomial provides a close approximation of $F(x)$ within the interval, especially near the interpolation points. However, due to Runge's phenomenon, the choice of points affects the accuracy—equidistant points are generally acceptable here but can lead to oscillations in wider intervals.

Practical Uses

- Numerical Integration: Using L4 polynomial to approximate integrals of $F(x)$ via polynomial quadrature.
- Function Approximation: Simplifying complex functions for real-time computation.
- Data Fitting: Modeling oscillatory behavior in physical or engineering systems.

Advanced Considerations

Chebyshev Nodes vs. Equidistant Points

For improved approximation and minimized oscillations, Chebyshev nodes are often preferred over equidistant points. They reduce the Runge phenomenon and yield more stable interpolants.

Extension to Higher Degrees

While degree-4 interpolation is manageable, increasing degree beyond this can lead to numerical instability. Chebyshev interpolation or spline methods are often employed for higher degrees or wider intervals.

Conclusion

Calculating the L4 polynomial for $F(x) = 6 \cos(x/2)$ over $[0.5, 1.0]$ exemplifies the core principles of polynomial interpolation. Through careful selection of data points, explicit evaluation, and basis polynomial construction, a high-fidelity approximation can be achieved. Such techniques underpin numerous computational methods, enabling scientists and engineers to approximate, analyze, and interpret complex functions with confidence.

The process underscores the delicate balance between approximation accuracy, computational complexity, and stability—central themes in numerical

Calculate L4 For F X 6cos X 2 Over 2 4 2 2 2 4 2 2

Calculate L4 For F X 6cos X 2 Over 2 4 2 2 2 4 2 2