Find The Exact Length Of The Curve. Y = Ln 1 X2 , 0 X 1 8

Find The Exact Length Of The Curve. Y = Ln(1 + X²), 0 ≤ X ≤ 8

Understanding how to determine the exact length of a curve is a fundamental problem in calculus, with applications ranging from physics to engineering. In this article, we will explore the process of finding the precise length of the curve defined by the function $y = \ln(1 + x^2)$ over the interval from $x = 0$ to $x = 8$. The goal is to derive a step-by-step solution, including the necessary calculus concepts, formulas, and calculations involved.

Fundamentals of Arc Length in Calculus

Before diving into the specific problem, it is essential to understand the general approach to calculating the length of a curve.

Arc Length Formula

Given a function $y = f(x)$, continuous and differentiable on the interval $[a, b]$, the exact length L of the curve from $x = a$ to $x = b$ is given by:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

This formula stems from the Pythagorean theorem, considering an infinitely small segment of the curve as a hypotenuse of a right triangle with differential changes in x and y.

Application to the Given Function

For the function $y = \ln(1 + x^2)$, our task is to:
1. Find the derivative $\frac{dy}{dx}$.
2. Plug this into the arc length formula.
3. Simplify the integrand.
4. Calculate the definite integral from 0 to 8.

Step 1: Compute the Derivative \(\frac{dy}{dx} \)

The function is:

\[
y = \ln(1 + x^2)
\]

Applying the chain rule:

\[
\frac{dy}{dx} = \frac{1}{1 + x^2} \times 2x = \frac{2x}{1 + x^2}
\]

This derivative expresses how \(y \) changes with respect to \(x \), which is essential for the arc length integral.

Step 2: Set Up the Arc Length Integral

Using the formula:

\[
L = \int_{0}^{8} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
\]

Substitute \(\frac{dy}{dx} = \frac{2x}{1 + x^2} \):

\[
L = \int_{0}^{8} \sqrt{1 + \left(\frac{2x}{1 + x^2} \right)^2} \, dx
\]

Simplify inside the square root:

\[
L = \int_{0}^{8} \sqrt{1 + \frac{4x^2}{(1 + x^2)^2}} \, dx
\]

To combine terms, write \(1 \) as \(\frac{(1 + x^2)^2}{(1 + x^2)^2} \):

\[
L = \int_{0}^{8} \sqrt{\frac{(1 + x^2)^2 + 4x^2}{(1 + x^2)^2}} \, dx
\]

Simplify numerator:

\[

(1 + x^2)^2 + 4x^2 = (1 + 2x^2 + x^4) + 4x^2 = 1 + 2x^2 + x^4 + 4x^2 = 1 + 6x^2 + x^4
\]

Thus,

\[
L = \int_{0}^{8} \frac{\sqrt{1 + 6x^2 + x^4}}{1 + x^2} \, dx
\]

Key observation: The numerator's radical resembles a perfect square.

Step 3: Simplify the Integrand

Let's analyze the numerator:

\[
\sqrt{1 + 6x^2 + x^4}
\]

Note that:

\[
(x^2 + 3)^2 = x^4 + 6x^2 + 9
\]

Compare with numerator:

\[
1 + 6x^2 + x^4
\]

which can be rewritten as:

\[
(x^2 + 3)^2 - 8
\]

since:

\[
(x^2 + 3)^2 = x^4 + 6x^2 + 9
\]

Subtract 8:

\[
x^4 + 6x^2 + 1 = (x^2 + 3)^2 - 8
\]

Thus,

$$
\sqrt{1 + 6x^2 + x^4} = \sqrt{(x^2 + 3)^2 - 8}
$$

The integrand becomes:

$$
\frac{\sqrt{(x^2 + 3)^2 - 8}}{1 + x^2}
$$

This expression suggests a substitution involving $x^2 + 3$.

Step 4: Substitution to Simplify the Integral

Let:

$$
t = x^2 + 3
$$

then:

$$
dt = 2x \, dx
$$

which implies:

$$
x \, dx = \frac{dt}{2}
$$

Note that as x goes from 0 to 8:

$$
t_{0} = 0^2 + 3 = 3
$$
$$
t_{8} = 8^2 + 3 = 64 + 3 = 67
$$

Express $1 + x^2$ in terms of t:

$$
1 + x^2 = (t - 3) + 1 = t - 2
$$

Express dx in terms of dt:

$$
x \, dx = \frac{dt}{2}
$$

But since $x = \sqrt{t - 3}$, then:

$$
dx = \frac{dt}{2x} = \frac{dt}{2 \sqrt{t - 3}}
$$

This complicates the substitution. Instead, observe that:

$$
\frac{\sqrt{t^2 - 8}}{t - 2}
$$

is the integrand after substitution, with the differential:

$$
dx = \frac{dt}{2x} = \frac{dt}{2 \sqrt{t - 3}}
$$

Alternatively, to avoid the complication, consider rewriting the integral in terms of t:

$$
L = \int_{x=0}^{8} \frac{\sqrt{(x^2 + 3)^2 - 8}}{1 + x^2} \, dx
$$

which can be expressed as:

$$
L = \int_{t=3}^{67} \frac{\sqrt{t^2 - 8}}{t - 2} \times \frac{dx}{dt} \, dt
$$

Given $t = x^2 + 3$, then:

$$
\frac{dt}{dx} = 2x \Rightarrow dx = \frac{dt}{2x}
$$

and $x = \sqrt{t - 3}$:

$$
dx = \frac{dt}{2 \sqrt{t - 3}}
$$

Substituting back into the integral:

\[
L = \int_{t=3}^{67} \frac{\sqrt{t^2 - 8}}{t - 2} \times \frac{1}{2 \sqrt{t - 3}} \, dt
\]

Simplify:

\[
L = \frac{1}{2} \int_{3}^{67} \frac{\sqrt{t^2 - 8}}{(t - 2) \sqrt{t - 3}} \, dt
\]

This integral is quite complex, and standard methods involve recognizing patterns or alternative substitutions.

Alternative Approach: Recognize the Derivative of the Original Function

Instead of overcomplicating, note that:

\[
\frac{dy}{dx} = \frac{2x}{1 + x^2}
\]

and observe that:

\[
\frac{2x}{1 + x^2} = \frac{d}{dx} \ln(1 + x^2)
\]

which aligns with the original function.

Given that, the integral simplifies to:

\[
L = \int_{0}^{8} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
\]

with the derivative known, perhaps a better approach is to express \(\sqrt{1 + \left(\frac{dy}{dx} \right)^2} \) directly in terms of \(x \).

Recall:

\[
\left(\frac{dy}{dx} \right)^2 = \left(

Frequently Asked Questions

How do you find the exact length of the curve y = ln(1 + x^2) from x = 0 to x = 8?

To find the length of the curve, use the formula L = ∫ from a to b √(1 + (dy/dx)^2) dx. Compute dy/dx for y = ln(1 + x^2), then evaluate the integral from 0 to 8.

What is the derivative dy/dx of y = ln(1 + x^2)?

The derivative dy/dx = (2x) / (1 + x^2).

How do you set up the integral for the arc length of y = ln(1 + x^2) over [0, 8]?

The arc length $L = \int_0^8 \sqrt{1 + [(2x)/(1 + x^2)]^2}\, dx$. Simplify the integrand before integrating.

Can the integral for the curve length be simplified before evaluation?

Yes, by simplifying the expression under the square root: 1 + (2x)^2 / (1 + x^2)^2 = ((1 + x^2)^2 + 4x^2) / (1 + x^2)^2. Simplify numerator to make the integral more manageable.

What is the final expression for the exact length of the curve y = ln(1 + x^2) from 0 to 8?

The exact length is $L = \int_0^8 \sqrt{[(1 + x^2)^2 + 4x^2]} / (1 + x^2)\, dx$, which can be further simplified and evaluated using substitution or numerical methods.

Additional Resources

Find The Exact Length Of The Curve. $Y = \ln(1 + x^2)$, $0 \le x \le 8$ – this is a classic calculus problem that combines the concepts of differentiation and integration to determine the precise length of a given curve over a specified interval. Accurately finding the length of a curve is crucial in various fields such as engineering, physics, and computer graphics, where the exact measurement of a path or shape influences design and analysis.

In this comprehensive guide, we will walk through the step-by-step process of calculating the exact length of the curve $Y = \ln(1 + x^2)$ from x = 0 to x = 8. We will explore the fundamental concepts involved, derive the necessary formulas, and perform the calculations meticulously to ensure clarity and accuracy.

Understanding the Problem

Before diving into calculations, it’s essential to understand what the problem entails:

- Curve Equation: $Y = \ln(1 + x^2)$
- Interval: x ranges from 0 to 8
- Objective: Find the exact length of the curve over this interval

The problem involves applying the formula for the arc length of a function, which is derived from the Pythagorean theorem extended to a curve.

The General Formula for Arc Length

The Arc Length Formula

For a continuous and differentiable function y = f(x) over an interval [a, b], the length L of the curve is given by:

$$ L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx $$

This formula essentially sums up infinitesimal straight-line segments along the curve, approximating the total length.

Application to Our Function

Since our function is $Y = \ln(1 + x^2)$, the first step is to compute its derivative dy/dx.

Step 1: Differentiating the Function

Given:

$$ y = \ln(1 + x^2) $$

Applying the chain rule:

$$ \frac{dy}{dx} = \frac{1}{1 + x^2} \times 2x = \frac{2x}{1 + x^2} $$

Step 2: Setting Up the Arc Length Integral

Using the formula:

$$ L = \int_{0}^{8} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx $$

Substitute dy/dx:

$$ L = \int_{0}^{8} \sqrt{1 + \left(\frac{2x}{1 + x^2}\right)^2} \, dx $$

Simplify inside the square root:

$$ L = \int_{0}^{8} \sqrt{1 + \frac{4x^2}{(1 + x^2)^2}} \, dx $$

Combine the terms under a common denominator:

$$ L = \int_{0}^{8} \sqrt{\frac{(1 + x^2)^2 + 4x^2}{(1 + x^2)^2}} \, dx $$

Step 3: Simplifying the Expression

Calculate numerator:

$$ (1 + x^2)^2 + 4x^2 = (1 + 2x^2 + x^4) + 4x^2 = 1 + 2x^2 + x^4 + 4x^2 $$

Combine like terms:

$$ 1 + (2x^2 + 4x^2) + x^4 = 1 + 6x^2 + x^4 $$

Now, write the integral as:

$$ L = \int_{0}^{8} \sqrt{\frac{1 + 6x^2 + x^4}{(1 + x^2)^2}} \, dx $$

Notice that numerator resembles a perfect square:

$$ 1 + 6x^2 + x^4 = (x^2 + 3)^2 $$

Verify:

$$ (x^2 + 3)^2 = x^4 + 6x^2 + 9 $$

Our numerator is $1 + 6x^2 + x^4$, which differs by 8 from this perfect square.
So, not a perfect square as-is, but perhaps expressible differently.

Alternatively, observe that:

$$ 1 + 6x^2 + x^4 = (x^2 + 3)^2 - 8 $$

Because:

$$ (x^2 + 3)^2 = x^4 + 6x^2 + 9 $$

Subtract 8:

$$ x^4 + 6x^2 + 1 = (x^2 + 3)^2 - 8 $$

So, the numerator is:

$$ (x^2 + 3)^2 - 8 $$

Therefore, the integral becomes:

$$ L = \int_{0}^{8} \frac{\sqrt{(x^2 + 3)^2 - 8}}{1 + x^2} \, dx $$

Step 4: Substituting to Simplify Further

Let’s attempt a substitution to simplify the integral:

Set:

$$ u = x^2 + 3 $$

Then,

$$ du/dx = 2x \Rightarrow dx = \frac{du}{2x} $$

But note that the substitution involves x, and the integral involves x in the denominator (1 + x^2), which can be expressed in terms of u:

$$ 1 + x^2 = u - 2 $$

Now, the limits:

- When x = 0, u = 0 + 3 = 3
- When x = 8, u = 64 + 3 = 67

Express x in terms of u:

$$ x = \sqrt{u - 3} $$

The differential:

$$ dx = \frac{1}{2\sqrt{u - 3}} du $$

The integral becomes:

$$
L = \int_{u=3}^{67} \frac{\sqrt{u^2 - 8}}{u - 2} \times \frac{1}{2\sqrt{u - 3}} du
$$

This looks complicated, but perhaps there's an easier approach.

Alternative Approach: Recognizing the Derivative Structure

Instead of complicating substitutions, notice that the derivative $dy/dx = 2x/(1 + x^2)$ can be rewritten:

$$
\frac{dy}{dx} = \frac{2x}{1 + x^2}
$$

which is a standard derivative with a known integral form.

Re-expressing the derivative:

$$
\frac{dy}{dx} = \frac{2x}{1 + x^2}
$$

This suggests an integral:

$$
\int \frac{2x}{1 + x^2} dx = \ln(1 + x^2) + C
$$

which is exactly our original function Y.

Step 5: Recognizing the Derivative and Simplifying the Arc Length Integral

Given that:

$$
\frac{dy}{dx} = \frac{2x}{1 + x^2}
$$

then:

$$
\left(\frac{dy}{dx}\right)^2 = \frac{4x^2}{(1 + x^2)^2}
$$

and

$$
1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{4x^2}{(1 + x^2)^2}
$$

which we previously simplified to:

$$
\frac{(1 + x^2)^2 + 4x^2}{(1 + x^2)^2} = \frac{1 + 6x^2 + x^4}{(1 + x^2)^2}
$$

As established earlier, numerator:

$$
1 + 6x^2 + x^4 = (x^2 + 3)^2 - 8
$$

So,

$$
L = \int_{0}^{8} \sqrt{\frac{(x^2 + 3)^2 - 8}{(1 + x^2)^2}} dx
$$

which can be written as:

$$
L = \int_{0}^{8} \frac{\sqrt{(x^2 + 3)^2 - 8}}{1 + x^2} dx
$$

Step 6: Numerical Approximation or Further Simplification

Given the complexity, an analytical solution may involve advanced techniques or recognizing special integrals. Alternatively, we can proceed with a substitution to evaluate numerically or look for a substitution that simplifies the radical.

Final Approach: Recognizing a Standard Form

Observe that:

$$
\sqrt{(x^2 + 3)^2 - 8} = \sqrt{(x^2 + 3)^2 - 8}
$$

which resembles the form:

$$
\sqrt{A^2 - B^2} = \sqrt{(A - B)(A + B)}
$$

But since $A = x^2 + 3$, $B = \sqrt{8}$, the radical becomes:

$$
\sqrt{(x^2 + 3)^2 - (\sqrt{8})^2}
$$

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