Help! Find The Exact Value Of Tan A In Simplest Radical Form.

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Understanding how to find the exact value of $\tan A$ in simplest radical form is a fundamental skill in trigonometry that can enhance your problem-solving abilities and deepen your grasp of geometric relationships. Whether you're a student preparing for exams or someone exploring the beauty of trigonometry, mastering this concept is essential. This article provides a comprehensive guide to understanding and calculating $\tan A$ in its simplest radical form, complete with step-by-step methods, examples, and tips.

What Is the Tangent Function?

Before diving into calculations, it's important to understand what the tangent function represents in trigonometry.

Definition of Tangent

- In a right-angled triangle, $\tan A$ is defined as the ratio of the length of the opposite side to the adjacent side:

$$
\tan A = \frac{\text{opposite}}{\text{adjacent}}
$$

- On the unit circle, $\tan A$ is the ratio of $\sin A$ to $\cos A$:

$$
\tan A = \frac{\sin A}{\cos A}
$$

Key Point: The tangent function is periodic and undefined at angles where $\cos A = 0$, such as 90° or $\frac{\pi}{2}$.

Methods to Find $\tan A$ in Simplest Radical Form

There are several approaches to determine $\tan A$ exactly, especially when dealing with special angles or given geometric information. The main methods include:

- Using special right triangles
- Applying the unit circle
- Utilizing known values and identities
- Employing inverse trigonometric functions with exact values

Let's explore each method in detail.

1. Using Special Right Triangles

Special right triangles, such as the 45-45-90 and 30-60-90 triangles, provide exact values for sine, cosine, and tangent at specific angles.

45-45-90 Triangle

- Angles: $45^\circ, 45^\circ, 90^\circ$
- Side lengths: $1, 1, \sqrt{2}$
- $\tan 45^\circ = \frac{1}{1} = 1$

Result:

$$
\boxed{\tan 45^\circ = 1}
$$

which is already in simplest radical form.

30-60-90 Triangle

- Angles: $30^\circ, 60^\circ, 90^\circ$
- Side lengths: $1, \sqrt{3}, 2$

Calculations:

- $\tan 30^\circ = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}}$

Expressed in simplest radical form:

$$
\boxed{\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}}
$$

- $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$

Summary:

Angle	$\tan A$ in radical form
30°	$\frac{1}{\sqrt{3}}$ or $\frac{\sqrt{3}}{3}$
45°	1
60°	$\sqrt{3}$

Tip: Recognize these special angles to quickly determine tangent values without calculator approximation.

2. Using the Unit Circle

The unit circle provides a geometric way to evaluate $\tan A$ at various angles, especially those not commonly covered by special triangles.

Steps:

1. Locate the angle A on the unit circle.
2. Determine the coordinates $(\cos A, \sin A)$.
3. Use the formula:

$$
\tan A = \frac{\sin A}{\cos A}
$$

4. Simplify the ratio to radical form, if necessary.

Example:

Find $\tan 135^\circ$:

- Coordinates: $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$

- Calculation:

$$

\tan 135^\circ = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1
\]

Result:

\[
\boxed{\tan 135^\circ = -1}
\]

This is already in simplest form.

Note: The signs of sine and cosine determine the quadrant and the sign of tangent.

3. Using Trigonometric Identities and Known Values

When given an angle or a problem involving complementary or supplementary angles, identities can help find \(\tan A\).

Common Identities:

- Reciprocal: \(\tan (90^\circ - A) = \cot A = \frac{1}{\tan A}\)
- Complementary angles: \(\tan (90^\circ - A) = \frac{1}{\tan A}\)
- Sum and Difference Formulas:

\[
\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\]

- Double Angle Formula:

\[
\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}
\]

Using these, if you know \(\tan A\) or can find \(\tan\) of related angles, you can solve for \(\tan A\).

Example:

Suppose \(\tan 2A = \sqrt{3}\), find \(\tan A\):

- Using the double angle formula:

$$
\sqrt{3} = \frac{2 \tan A}{1 - \tan^2 A}
$$

- Cross-multiplied:

$$
\sqrt{3}(1 - \tan^2 A) = 2 \tan A
$$

- Simplify:

$$
\sqrt{3} - \sqrt{3} \tan^2 A = 2 \tan A
$$

- Rearranged as quadratic in $\tan A$:

$$
\sqrt{3} \tan^2 A + 2 \tan A - \sqrt{3} = 0
$$

- Use quadratic formula:

$$
\tan A = \frac{-2 \pm \sqrt{(2)^2 - 4 \times \sqrt{3} \times (-\sqrt{3})}}{2 \sqrt{3}}
$$

- Simplify discriminant:

$$
4 - 4 \times \sqrt{3} \times (-\sqrt{3}) = 4 + 4 \times 3 = 4 + 12 = 16
$$

- Final solutions:

$$
\tan A = \frac{-2 \pm \sqrt{16}}{2 \sqrt{3}} = \frac{-2 \pm 4}{2 \sqrt{3}}
$$

- Two possible solutions:

1. $\frac{-2 + 4}{2 \sqrt{3}} = \frac{2}{2 \sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

2. $\frac{-2 - 4}{2 \sqrt{3}} = \frac{-6}{2 \sqrt{3}} = -\frac{3}{\sqrt{3}} = -\sqrt{3}$

Thus, $\tan A$ could be $\frac{\sqrt{3}}{3}$ or $-\sqrt{3}$, depending on the context.

Expressing $\tan A$ in Simplest Radical Form

When aiming to write $\tan A$ in simplest radical form, consider these tips:

- Rationalize denominators when necessary. For example:

$$
\frac{1}{\sqrt{3}} \quad \text{becomes} \quad \frac{\sqrt{3}}{3}
$$

- Simplify radicals where possible, e.g.,

$$
\sqrt{12} = \sqrt{4 \times 3} = 2 \sqrt{3}
$$

- Keep the radical in numerator or denominator only if it simplifies the expression.

Examples of simplification:

Expression	Simplified Radical Form
$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{3}}{3}$
$\frac{\sqrt{12}}{4}$	$\frac{2 \sqrt{3}}{4} = \frac{\sqrt{3}}{2}$
$\frac{\sqrt{50}}{5}$	$\frac{5 \sqrt{2}}{5} = \sqrt{2}$

Practice Problems and Solutions