Prove That No Group Can Have Exactly Two Elements Of Order 2.

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Introduction

Groups are fundamental algebraic structures that capture the concept of symmetry and operation within a set. One interesting aspect of groups involves the elements of order 2, i.e., elements g such that $g^2 = e$, where e is the identity element. A natural question arises: can a group possess exactly two elements of order 2? Surprisingly, the answer is no, and this article aims to rigorously prove that no such group exists. We will explore the underlying principles, leverage key theorems, and develop a comprehensive understanding of why this is the case.

Preliminaries and Basic Definitions

Before delving into the proof, it's essential to clarify the foundational concepts involved.

Definitions

- **Group:** A set G equipped with an operation $\ast$ satisfying closure, associativity, the existence of an identity element e, and the existence of inverse elements for all members.
- **Order of an element:** For $g \in G$, the order $\text{ord}(g)$ is the smallest positive integer n such that $g^n = e$. If no such n exists, g has infinite order.
- **Element of order 2:** An element $g \neq e$ such that $g^2 = e$ and $\text{ord}(g) = 2$.

Key Observations

- The identity element e always satisfies $e^1 = e$, so its order is 1.
- Elements of order 2 are their own inverses since $g^2 = e$ implies $g = g^{-1}$.
- The set of elements of order 2 often exhibits special properties within the group.

Statement of the Theorem

Theorem: There does not exist a group G such that the set of elements of order 2 contains exactly two elements.

In symbols, if G is a group, then

$$
|\{ g \in G \setminus \{ e \} : g^2 = e \}| \neq 2.
$$

This means that the number of elements of order 2 in any group is never exactly two.

Outline of the Proof Strategy

Our approach involves the following steps:

1. Assume for contradiction that such a group G exists with exactly two elements of order 2.
2. Analyze the structure of the set of elements of order 2, including their interactions with other elements.
3. Leverage the properties of conjugation and group actions to derive contradictions.
4. Conclude that such a group cannot exist.

Detailed Proof

Step 1: Assume Existence of Such a Group

Suppose there exists a group G with exactly two elements of order 2, say a and b, both different from the identity e. Thus,

$$
\{ g \in G : g^2 = e, g \neq e \} = \{ a, b \}.
$$

Note that:

- $a^2 = e$,
- $b^2 = e$,
- $a \neq b$,
- $a, b \neq e$.

Step 2: Explore the Behavior of These Elements

Since both a and b are of order 2, they are their own inverses:

$$
a^{-1} = a, \quad b^{-1} = b.
$$

Now, consider the product ab:

- The element ab may or may not be of order 2.
- Our goal is to analyze the properties of ab, ba, and related elements.

Step 3: Properties of the Product ab

Let's examine the order of ab:

- Compute $(ab)^2$:

$$
(ab)^2 = (ab)(ab) = a b a b.
$$

- Since $a^2 = e$ and $b^2 = e$, and elements of order 2 are their own inverses, we analyze the expression:

\[

$$
a b a b.
$$

- Note that:

$$
a b a b = a (b a) b.
$$

- If a and b commute ($ab = ba$), then:

$$
a b a b = a a b b = e e = e,
$$

so $(ab)^2 = e$, implying ab has order 1 or 2.

- Since $ab \neq e$ (unless $a = b$, but we assumed $a \neq b$), then:

$$
\text{ord}(ab) = 2,
$$

and thus ab is also an element of order 2.

- Important conclusion: If a and b commute, then ab is of order 2.

On the other hand, what if a and b do not commute? We need to analyze the implications.

Step 4: Conjugation and the Set of Elements of Order 2

Suppose a and b do not commute. Consider the conjugate of a by b:

$$
b a b^{-1}.
$$

Since b has order 2, $b^{-1} = b$. Therefore,

$$
b a b.
$$

Now, examine whether this conjugate has order 2:

$$
(b a b)^2 = b a b b a b = b a e a b = b a a b = b e b = b b = e,
$$

because $a^2 = e$ and $b^2 = e$.

Thus,

$$
(b a b)^2 = e,
$$

which implies $b a b$ is of order 2.

Key point: conjugation by any element preserves the order of elements, and in particular, the conjugate of an element of order 2 remains of order 2.

Step 5: The Set of Elements of Order 2 is Closed Under Conjugation

From the previous step, the conjugate of a by b, which is $b a b$, is of order 2.

- Since we only have exactly two elements of order 2, a and b, the conjugation $b a b$ must be either a or b. Otherwise, we would have more than two elements of order 2.

- Therefore, either:

1. $b a b = a$, or

2. $b a b = b$.

Let's analyze these cases.

Step 6: Case 1 - $b a b = a$

If $b a b = a$, then:

$$
b a = a b,
$$

which implies a and b commute.

- As previously discussed, if a and b commute, then:

$$
(ab)^2 = e,
$$

and ab is of order 2.

- But this would mean that the subgroup generated by a and b contains at least three elements of order 2: a, b, and ab.

- Since we assumed only two elements of order 2 exist, this can only happen if $ab = a$ or $ab = b$.

- But:

$$
ab = a \implies b = e,
$$
which contradicts $b \neq e$.

Similarly,

$$
ab = b \implies a = e,
$$
which contradicts $a \neq e$.

- Therefore, the assumption $b a b = a$ and the commutativity of a, b lead to contradictions unless $a = b$, which is false.

Case 2 - $bab = b$

Similarly, suppose $bab = b$:

$$
bab = b \implies a = e,
$$
which contradicts that $a \neq e$.

Conclusion from both cases: the only consistent scenario is that the conjugation of a by b swaps them, i.e.,

$$
bab = c,
$$

where c is an element of order 2 different from both a and b.

However, our assumption is that the set of elements of order 2 contains exactly a and b. Therefore, bab must be either a or \(b\

Frequently Asked Questions

Why can't a group have exactly two elements of order 2?

Because in any group, the set of elements of order 2, along with the identity, forms a subgroup called the Klein four-group or a subset that violates group properties if only two elements of order 2 exist. This leads to contradictions, making it impossible to have exactly two elements of order 2.

What is the significance of the element of order 2 in a group?

An element of order 2 is its own inverse, meaning it satisfies $g^2 = e$. The presence and number of such elements influence the structure of the group, but having exactly two such elements leads to contradictions, as shown in the proof.

How does the proof that no group can have exactly two elements of order 2 relate to subgroup theory?

The proof involves analyzing the set of elements of order 2 and their interactions, demonstrating that this set must either be empty, contain exactly one element, or form a subgroup isomorphic to the Klein four-group if it contains more than one element, thus excluding the possibility of exactly two.

Can you give an outline of the proof that no group has exactly two elements of order 2?

Yes. The proof considers the set of all elements of order 2. If there are more than one, their pairwise products also have order 2 or other properties that imply the set forms a subgroup. This subgroup must be either trivial, cyclic of order 2, or isomorphic to Klein four-group, but never just two elements of order 2, leading to the conclusion that exactly two such elements cannot exist.

Are there known examples of groups with 0 or more than 2 elements of order

2?

Yes. The trivial group has zero elements of order 2, and the Klein four-group has exactly three elements of order 2. Many dihedral and symmetric groups also contain multiple elements of order 2, but none have exactly two.

What is the broader algebraic significance of the fact that no group has exactly two elements of order 2?

This fact highlights the structural constraints within groups and illustrates how subgroup properties and element orders impose limitations on possible configurations, enriching our understanding of group classification and symmetry structures.

Additional Resources

Prove That No Group Can Have Exactly Two Elements Of Order 2

Introduction

Group theory, a fundamental branch of abstract algebra, explores algebraic structures known as groups. These structures encapsulate the idea of symmetry, operation, and the algebraic properties that arise from combining elements within a set. One of the key features studied in groups is the order of elements, which describes how many times an element must be combined with itself to reach the identity element.

A particularly interesting question in the structure theory of groups concerns the possible configurations of elements of order 2—elements g such that $g^2 = e$, where e is the identity element. Specifically, the question arises: Can a group have exactly two elements of order 2? This article provides a comprehensive proof that the answer is negative, exploring the underlying algebraic principles, illustrating key concepts with examples, and examining the implications of this result.

Background Concepts and Definitions

Before delving into the proof, it is essential to clarify some foundational concepts:

- Group: A set G equipped with a binary operation $\cdot$ satisfying closure, associativity, the existence of an identity element e, and the existence of inverses for all elements.

- Order of an element: For $g \in G$, the order $|g|$ is the smallest positive integer n such that $g^n = e$. If no such n exists, g is said to have infinite order.

- Elements of order 2: Elements g where $g^2 = e$ and $g \neq e$. These are their own inverses, i.e., $g = g^{-1}$.

- Involution: An element of order 2 is called an involution.

The Core Question: Can a Group Have Exactly Two Elements of Order 2?

Intuitively, one might suspect that it is possible for a group to have precisely two involutions. For example, the Klein four-group $V_4 \cong C_2 \times C_2$ has three involutions besides the identity. But can the number of involutions be exactly two?

The answer, as we shall demonstrate, is a decisive no. The proof hinges on the algebraic properties of involutions and the structure of groups containing multiple elements of order 2.

The Fundamental Theorem and Key Observations

The proof relies on the following key observations:

1. Involutions form conjugacy classes: In a group G, the set of involutions can form one or multiple conjugacy classes depending on the group's structure.

2. The set of all involutions: Denote by I the set of all involutions in G. It is known that $I \cup \{ e \}$ is closed under conjugation and, in some cases, under certain operations, depending on the group's properties.

3. Behavior under conjugation: In particular, in groups where involutions exist, they often exhibit symmetry properties, especially in relation to the group's center and normal subgroups.

The Main Theorem and Its Proof

Theorem: No group can have exactly two elements of order 2.

Proof:

Suppose, for contradiction, that there exists a group G with exactly two elements of order 2, say a and b. Both a and b are involutions, so:

$$
a^2 = e, \quad b^2 = e, \quad a \neq e, \quad b \neq e
$$

and the set of elements of order 2 is $\{ a, b \}$.

Our goal is to demonstrate that this assumption leads to a contradiction, implying such a group cannot exist.

Step 1: Consider the product ab

Since a and b are involutions, analyze the element ab.

- Compute $(ab)^2$:

$$
(ab)^2 = abab
$$

- Since $a^2 = e$ and $b^2 = e$:

$$
abab = a (b a) b
$$

- Now, examine $b a$:

Because a and b are involutions, the behavior of a and b when multiplied depends on whether they commute.

Step 2: Investigate whether a and b commute

There are two cases:

Case 1: a and b commute

If $a b = b a$, then:

$$
(ab)^2 = a b a b = a a b b = e e = e
$$

Thus, ab is an involution.

- Since $a \neq e$ and $b \neq e$, and $a \neq b$, then:

$$
ab \neq e
$$

because:

$$
(ab) = e \implies a = b
$$

which contradicts the assumption that $a \neq b$. Therefore, $ab \neq e$, but $(ab)^2 = e$, which implies ab is an involution.

- But this involution ab is distinct from a and b (since $a \neq b$, and neither equals e).

- Therefore, the set of involutions contains at least three elements: a, b, and ab.

This contradicts the initial assumption that there are exactly two involutions in G.

Case 2: a and b do not commute

Suppose $a b \neq b a$. Then:

$$
(ab)^2 = a b a b
$$

- Rewrite as:

$$
a b a b
$$

- Use the relation:

$$
a b a = a (b a) = a (a b) \quad \text{(since } a \text{ and } b \text{ may not commute)}
$$

But more straightforwardly, note that:

$$
a b a = (a a) b = e b = b
$$

Wait, this is incorrect unless a and b commute. So, in the non-commuting case, $a b a \neq b$ in general.

Alternatively, compute:

$$
(ab)^2 = a b a b
$$

which is equal to:

$$
a (b a) b
$$

Now, this expression simplifies differently depending on whether a and b commute.

In general, for non-commuting involutions:

$$
a b a b \neq e
$$

and, more specifically, the order of ab can be greater than 2. But crucially, the key point is that regardless of whether a and b commute, the element ab is either:

- an involution (if $(ab)^2 = e$)
- or an element of order greater than 2.

Step 3: Summarize implications

From Case 1, if a and b commute, then ab is an involution distinct from a and b, so the total number of involutions is at least three, contradicting the initial assumption.

From Case 2, if a and b do not commute, then ab is not an involution, and the set of elements of order 2 remains just a and b.

But now, the key question is: can the product $a b$ have order greater than 2? If so, then the set of involutions remains just a and b. However, in many groups, the presence of multiple involutions leads to the formation of larger subgroups, particularly when involutions commute.

Step 4: The role of the subgroup generated by a and b

Consider the subgroup $H = \langle a, b \rangle$. There are two possibilities:

- H is isomorphic to $C_2 \times C_2$:

- This subgroup contains exactly three involutions: e, a, b, and $a b$.

- Since a and b commute, this is the Klein four-group, which has three involutions, not two.

- H is cyclic or dihedral:

- If a and b do not commute, then H is a dihedral group or other non-abelian group containing multiple involutions.

In either case, the number of involutions in H is at least three (including the identity), which implies that G contains at least those three involutions.

Conclusion of the proof:

- If a group G has two involutions a and b, then the subgroup $\langle a, b \rangle$ either:

- Is is

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