Solve The Following Initial Value Problem. Y''(t)=18t-84t^5

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When tackling differential equations, particularly second-order differential equations with initial conditions, understanding the process to find the particular solution is essential. The problem presented—Y''(t) = 18t - 84t^5—serves as a classic example of a nonhomogeneous second-order differential equation. This article provides a comprehensive step-by-step guide to solving this initial value problem, ensuring clarity for learners and practitioners alike.

Understanding the Problem and Its Components

Before diving into the solution, it’s crucial to understand the nature of the differential equation and what is required.

The Differential Equation

- Equation: Y''(t) = 18t - 84t^5

- Type: Nonhomogeneous second-order ordinary differential equation (ODE)

- Goal: Find the general solution Y(t), and if initial conditions are provided, determine the particular solution.

Initial Conditions

While the problem statement mentions an initial value problem, specific initial conditions (such as Y(0) = y_0 and Y'(0) = y'_0) are necessary for a complete solution. For demonstration purposes, assume initial conditions:

- Y(0) = 0

- Y'(0) = 0

If actual initial conditions are different, they can be substituted accordingly.

Step 1: Find the General Solution to the

Homogeneous Equation

The first step involves solving the associated homogeneous differential equation:

$Y''(t) = 0$

This simplifies the process by focusing on the homogeneous part before adding particular solutions.

Homogeneous Equation Solution

- The homogeneous equation:

$Y''(t) = 0$

- Integrating twice:

1. Integrate $Y''(t) = 0$ to find $Y'(t)$:

$Y'(t) = C_1$

2. Integrate $Y'(t)$ to find $Y(t)$:

$Y(t) = C_1 t + C_2$

- Homogeneous solution:

$$
Y_h(t) = C_1 t + C_2
$$

where C_1 and C_2 are arbitrary constants.

Step 2: Find a Particular Solution to the Nonhomogeneous Equation

Given the nonhomogeneous term (right side of the differential equation), $18t - 84t^5$, the method of undetermined coefficients or variation of parameters can be used. Since the right side is a polynomial, the method of undetermined coefficients is suitable.

Choosing a Trial Solution

- The right side is a polynomial of degree 5.

- The particular solution $Y_p(t)$ should be a polynomial of degree 5:

$$
Y_p(t) = A t + B t^2 + C t^3 + D t^4 + E t^5
$$

Note: Since the second derivative of a polynomial reduces the degree by 2, the chosen polynomial must account for the derivatives to match the right side.

Calculating Derivatives of $Y_p(t)$

- First derivative:

$$
Y_p'(t) = A + 2 B t + 3 C t^2 + 4 D t^3 + 5 E t^4
$$

- Second derivative:

$$
Y_p''(t) = 2 B + 6 C t + 12 D t^2 + 20 E t^3
$$

Substitute $Y_p''(t)$ into the Differential Equation

$$
Y_p''(t) = 18 t - 84 t^5
$$

Matching the derivatives:

$$
2 B + 6 C t + 12 D t^2 + 20 E t^3 = 18 t - 84 t^5
$$

Since the right side contains a t^5 term, but the second derivative of a polynomial of degree 5 will be degree 3, it indicates we need to include a polynomial term of degree 5 in $Y_p(t)$ to account for the t^5 term.

Note: We observe that the second derivative of a degree 5 polynomial yields a degree 3 polynomial, which cannot produce a t^5 term directly. Therefore, to match the t^5 term, the particular solution must include a polynomial of degree 7 or higher. But a more straightforward approach is to recognize that the second derivative of a polynomial of degree n is degree (n-2). Since the RHS includes t^5, the particular solution must include a t^7 term.

Alternative approach:

Given the RHS includes polynomial terms up to degree 5, and second derivatives of polynomials of degree 7 will produce degree 5 terms, so:

$$
Y_p(t) = a t^7 + \text{lower degree terms}
$$

But to avoid unnecessary complexity, a better approach is to treat the particular solution as a sum of polynomials matching the RHS:

$$
Y_p(t) = P(t) = A t + B t^2 + C t^3 + D t^4 + E t^5
$$

and accept that the derivatives will produce degrees less than or equal to 3, which may not match the RHS's t^5 term. To resolve this, use the method of variation of parameters or polynomial ansatz of higher degree.

Simplified approach:

Since the RHS is polynomial, and the derivatives of polynomial solutions are polynomial, the second derivative of a polynomial of degree n is degree n-2. To generate a t^5 term after taking the second derivative, the particular solution must be at least degree 7:

$$
Y_p(t) = a t^7 + \text{lower degree terms}
$$

Calculating the second derivative:

$$
Y_p''(t) = 42 a t^5 + \text{lower degree terms}
$$

Matching the t^5 term:

$$
42 a t^5 = -84 t^5 \Rightarrow 42 a = -84 \Rightarrow a = -2
$$

Thus, the particular solution includes:

$$
Y_p(t) = -2 t^7 + \text{terms of degree 5 and below}
$$

But since the RHS only includes t and t^5, the lower degree terms can be adjusted accordingly.

Final particular solution:

$$
Y_p(t) = -2 t^7 + P_5(t)
$$

where $P_5(t)$ is a polynomial of degree 5 to match the other terms.

Given the complexity, a more straightforward approach is to directly apply the method of undetermined coefficients for each term:

- For the term 18t:

The particular solution for this term is proportional to t, as the second derivative of a quadratic term.

- For the term -84 t^5:

The particular solution is proportional to t^7, as explained.

Therefore, the particular solution takes the form:

$$
Y_p(t) = a t + b t^3 + c t^5 + d t^7
$$

Calculating derivatives:

$$
Y_p'(t) = a + 3 b t^2 + 5 c t^4 + 7 d t^6
$$

$$
Y_p''(t) = 6 b t + 20 c t^3 + 42 d t^5
$$

Now, substitute into the differential equation:

$$
Y_p''(t) = 18 t - 84 t^5
$$

which gives:

$$
6 b t + 20 c t^3 + 42 d t^5 = 18 t - 84 t^5
$$

Matching coefficients of like powers:

- For t:

$$
6 b t = 18 t \Rightarrow 6 b = 18 \Rightarrow b = 3
$$

\]

- For t^3:

\[
20 c t^3 = 0 \Rightarrow c = 0
\]

- For t^5:

\[
42 d t^5 = -84 t^5 \Rightarrow 42 d = -84 \Rightarrow d = -2
\]

Now, the particular solution:

\[
Y_p(t) = a t + 0 \cdot t^3 + 3 t^5 - 2 t^7
\]

The coefficient a remains undetermined because it does not appear in the derivatives; for the particular solution, a can be set to zero, as it corresponds to the homogeneous solution.

Final particular solution:

\[
Y_p(t) = 3 t^5 - 2 t^7
\]

Summary of the particular solution:

\[
\boxed{
Y_p(t) = 3 t^5 - 2 t^7
}
\]

Note:

- The homogeneous solution is \(Y_h(t) = C_1 t + C_2 \)
- The particular solution is \(Y_p(t) = 3 t^5 - 2 t^7 \)

Step 3: Write the General Solution

Combining the homogeneous and particular solutions:

\[
$Y(t) = C_1 t$

Frequently Asked Questions

What is the differential equation given in the initial value problem?

The differential equation is $y''(t) = 18t - 84t^5$.

How do you find the first step to solve the initial value problem $y''(t) = 18t - 84t^5$?

The first step is to integrate $y''(t)$ once with respect to t to find $y'(t)$.

What is the integral of $y''(t) = 18t - 84t^5$ to find $y'(t)$?

Integrate term-by-term: $y'(t) = \int(18t - 84t^5)\,dt = 9t^2 - 14t^6 + C1$, where C1 is a constant of integration.

How do you find y(t) after finding y'(t)?

Integrate $y'(t) = 9t^2 - 14t^6 + C1$ with respect to t to get y(t).

What is the expression for y(t) after integrating y'(t)?

$y(t) = 3t^3 - 2t^7 + C1\,t + C2$, where C2 is another constant of integration.

If initial conditions are provided, how do they help in solving this initial value problem?

They allow you to solve for the constants C1 and C2 by substituting the initial values of y(t) and y'(t) at a specific t.

Can you summarize the general solution to the initial value problem?

Yes, the general solution is $y(t) = 3t^3 - 2t^7 + C1\,t + C2$, with C1 and C2 determined by initial conditions.

Are there any special considerations when integrating to solve such differential equations?

Yes, ensure proper integration of polynomial terms and account for the constants of integration; also, verify initial conditions to find specific solutions if provided.

What are the key steps to solving the initial value problem y''(t) = 18t - 84t^5?

The key steps are: integrate y''(t) to find y'(t), then integrate y'(t) to find y(t), and finally apply any initial conditions to determine constants.

Additional Resources

Solve The Following Initial Value Problem. Y''(t)=18t-84t^5

Introduction

In the realm of differential equations, initial value problems (IVPs) serve as fundamental tools for modeling real-world phenomena where initial conditions are known. They are crucial in physics, engineering, biology, and numerous scientific disciplines, providing insights into how systems evolve over time. One particular class of IVPs involves second-order differential equations, which describe systems with acceleration or curvature, such as oscillatory systems, mechanical vibrations, and thermal conduction processes.

The specific IVP under consideration, Y''(t) = 18t - 84t^5, presents a second-order ordinary differential equation (ODE) with a polynomial right-hand side. The goal is to find a function $Y(t)$ that satisfies this differential equation, along with given initial conditions $Y(t_0) = Y_0$ and $Y'(t_0) = Y'_0$. Although the initial conditions are not specified in the prompt, the general solution process can be demonstrated, and the approach can be tailored once initial conditions are provided.

This article offers a comprehensive, detailed examination of solving this particular IVP. We will explore the mathematical background, solution methodology, step-by-step integration, and interpret the implications of the solution within a broader context.

Mathematical Background and Significance

Understanding Second-Order Differential Equations

A second-order differential equation involves the second derivative of an unknown function, denoted as $Y''(t)$. It takes the general form:

$$
Y''(t) = f(t, Y(t), Y'(t))
$$

In many classical mechanics problems, this relates to forces, acceleration, or curvature. When the right side, $f(t)$, depends solely on the independent variable t, the equation is called a nonhomogeneous linear ODE with constant coefficients, or more generally, a nonhomogeneous ODE if the right side varies with t.

Polynomial Forcing Terms

The right-hand side in this problem, $18t - 84t^5$, is a polynomial function in t. Polynomial forcing terms are common in physics, representing, for example, external forces that vary with time. The structure simplifies integration since polynomial functions are straightforward to integrate, and their derivatives are well-understood.

Importance of Initial Conditions

To obtain a unique solution, initial conditions specify the value of the function and its first derivative at a particular point $t = t_0$. These conditions determine the constants of integration that emerge during the solution process and ensure the solution aligns with physical or practical constraints.

Step-by-Step Solution Approach

1. Recognizing the Type of Equation

The differential equation:

$$
Y''(t) = 18t - 84t^5
$$

is a second-order, nonhomogeneous, linear ODE with respect to $Y(t)$. Since the right side depends only on t, the general solution can be obtained via double integration of the right side, considering the indefinite integral and then adding the general solution of the associated homogeneous equation.

2. Solving the Homogeneous Equation

The related homogeneous equation:

$$
Y''(t) = 0
$$

has the general solution:

$$
Y_h(t) = C_1 t + C_2
$$

where C_1 and C_2 are arbitrary constants, determined by initial conditions.

3. Integrating the Nonhomogeneous Term

Given that $Y''(t) = f(t)$, and knowing that:

$$
Y'(t) = \int Y''(t) dt + D
$$
$$
Y(t) = \int Y'(t) dt + E
$$

we can find $Y(t)$ by integrating $f(t)$ twice, adding integration constants at each step.

Detailed Integration Process

First Integration: Computing $Y'(t)$

Given:

$$
Y''(t) = 18t - 84t^5
$$

Integrate both sides with respect to t:

$$
Y'(t) = \int (18t - 84t^5) dt + D
$$

Calculating the integral:

$$
\int 18t dt = 18 \times \frac{t^2}{2} = 9 t^2
$$
$$
\int -84 t^5 dt = -84 \times \frac{t^6}{6} = -14 t^6
$$

Thus, the expression for $Y'(t)$:

$$
Y'(t) = 9 t^2 - 14 t^6 + D
$$

where D is an arbitrary constant.

Second Integration: Computing $Y(t)$

Now, integrate $Y'(t)$:

$$
Y(t) = \int (9 t^2 - 14 t^6 + D) dt + E
$$

Breaking into parts:

$$
\int 9 t^2 dt = 9 \times \frac{t^3}{3} = 3 t^3
$$
$$
\int -14 t^6 dt = -14 \times \frac{t^7}{7} = -2 t^7
$$
$$
\int D dt = D t
$$

Therefore, the general solution:

$$
Y(t) = 3 t^3 - 2 t^7 + D t + E
$$

where D and E are arbitrary constants to be determined via initial conditions.

Complete General Solution

Combining the homogeneous and particular solutions, the comprehensive general solution is:

$$
\boxed{
Y(t) = C_1 t + C_2 + 3 t^3 - 2 t^7 + D t + E
}
$$

But because the particular solution already includes the linear terms $D t + E$, the complete solution simplifies to:

$$
Y(t) = C_1 t + C_2 + 3 t^3 - 2 t^7
$$

where C_1, C_2 are arbitrary constants combining the previous integration constants.

Note: In the context of solving the initial value problem, the constants C_1 and C_2 are determined from initial conditions, typically specified as $Y(t_0) = Y_0$ and $Y'(t_0) = Y'_0$.

Application of Initial Conditions

Suppose initial conditions are provided, such as:

$$
Y(t_0) = Y_0
$$
$$
Y'(t_0) = Y'_0
$$

then, substituting these into the solution:

$$
Y(t_0) = C_1 t_0 + C_2 + 3 t_0^3 - 2 t_0^7
$$
$$
Y'(t_0) = C_1 + 9 t_0^2 - 14 t_0^6
$$

allows us to solve for C_1 and C_2:

$$
C_1 = Y'_0 - 9 t_0^2 + 14 t_0^6
$$
$$
C_2 = Y_0 - C_1 t_0 - 3 t_0^3 + 2 t_0^7
$$

Once these constants are determined, the explicit solution $Y(t)$ is fully specified.

Physical and Practical Implications

Modeling Mechanical Systems

This solution process is akin to analyzing a mechanical system subjected to external forces modeled by polynomial functions. The particular solution reflects the system's response to these external inputs, while the homogeneous part captures natural vibrations or free responses.

Significance of Polynomial Forcing

Polynomials are often used to approximate complex external influences due to their simplicity and mathematical tractability. The explicit form of the solution reveals how the system's displacement evolves under such forces, which can be invaluable in engineering design, control systems, and physics.

Sensitivity to Initial Conditions

The arbitrary constants underscore the importance of initial conditions in predicting the system's future behavior. Small variations in initial position or velocity can significantly influence the evolution, emphasizing the need for precise measurements in practical applications.

Broader Context and Extensions

Nonlinear and Variable Coefficient Equations

While the current problem involves a linear, constant coefficient differential equation, many real-world problems are nonlinear or have variable coefficients, complicating solution strategies. Techniques such as variation of parameters, Frobenius method, or numerical methods become necessary.

Numerical Solutions and Approximate Methods

In cases where the forcing term isn't a polynomial or the equation is more complex, numerical methods like Runge-Kutta or finite difference schemes are employed to approximate solutions, especially when initial conditions are specified.

Higher-Order and Systems of Equations

The approach demonstrated here extends to higher-order differential equations and systems of equations, which often appear in advanced modeling scenarios. Analytical solutions may be more challenging, but the fundamental principles remain similar.

Conclusion

The process of solving the initial value problem $Y''(t) = 18t - 84t^5$ exemplifies classical techniques in differential equations—integrating polynomial functions twice, incorporating initial conditions, and interpreting the resulting solution within a physical or engineering context. The explicit solution derived:

\[
Y(t) = C

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