There Can Be A Graph With Degree Sequence 5,4,3,3,2,2,1 T/F?

There Can Be A Graph With Degree Sequence 5,4,3,3,2,2,1 T/F?

Understanding whether a particular degree sequence can be realized as a simple graph is a fundamental question in graph theory. Degree sequences are sequences of non-negative integers that represent the degrees of vertices in a graph. They are central to many areas, including network analysis, combinatorics, and computer science.

In this article, we explore the degree sequence 5, 4, 3, 3, 2, 2, 1 and investigate whether such a sequence can be realized by a simple, undirected graph. We will delve into the theoretical background, apply classic criteria such as the Havel-Hakimi algorithm and the Erdős–Gallai theorem, and discuss practical steps to determine the realizability of this sequence.

Understanding Degree Sequences in Graph Theory

Before addressing the specific sequence, it's essential to understand what degree sequences are and the conditions under which they can be realized as simple graphs.

What Is a Degree Sequence?

A degree sequence is an ordered list of the degrees (number of edges incident to each vertex) of the vertices in a graph. For example, if a graph has vertices with degrees 5, 4, 3, 3, 2, 2, and 1, then the degree sequence is (5, 4, 3, 3, 2, 2, 1).

Key Points:

- The degrees are non-negative integers.
- The sequence can be sorted in non-increasing order for convenience.
- Not all sequences of integers can be realized as simple graphs.

Conditions for Graphical Degree Sequences

A degree sequence is called graphical if there exists a simple graph whose degrees of vertices match the sequence. Several criteria help determine whether a sequence is graphical:

1. Handshaking Lemma:
The sum of degrees must be even because each edge contributes 2 to the total degree sum.

2. Havel-Hakimi Algorithm:
A constructive process to test if a sequence is graphical.

3. Erdős–Gallai Theorem:

Provides necessary and sufficient conditions based on inequalities involving partial sums of the degree sequence.

Applying These Criteria to the Sequence 5, 4, 3, 3, 2, 2, 1

Let's analyze the sequence step-by-step.

Step 1: Sort the Sequence

Arrange the sequence in non-increasing order:

(5, 4, 3, 3, 2, 2, 1)

Step 2: Check the Handshaking Lemma

Calculate the sum:

5 + 4 + 3 + 3 + 2 + 2 + 1 = 20

Since 20 is even, the sequence passes the necessary condition for being graphical.

Step 3: Apply the Havel-Hakimi Algorithm

The Havel-Hakimi algorithm proceeds as follows:

1. Order the sequence in non-increasing order.
2. Remove the first element (largest degree).
3. Subtract 1 from the next 'd' elements, where 'd' is the removed degree.
4. Repeat with the new sequence until all elements are zero or the sequence becomes invalid.

Let's perform the steps:

Initial sequence: (5, 4, 3, 3, 2, 2, 1)

- Remove 5, subtract 1 from the next 5 elements:

Remaining sequence before subtraction: (4, 3, 3, 2, 2, 1)

Subtract 1 from the first five elements:

- 4 → 3
- 3 → 2

- 3 → 2
- 2 → 1
- 2 → 1

Sequence after subtraction: (3, 2, 2, 1, 1, 1)

- Sort: (3, 2, 2, 1, 1, 1)

Repeat:

- Remove 3, subtract 1 from next 3 elements:

Remaining sequence: (2, 2, 1, 1, 1)

Subtract 1 from first three:

- 2 → 1
- 2 → 1
- 1 → 0

Sequence: (1, 1, 0, 1)

Sort: (1, 1, 1, 0)

Repeat:

- Remove 1, subtract 1 from next 1 element:

Remaining sequence: (1, 1, 0)

Subtract 1 from first element:

- 1 → 0

Sequence: (1, 0)

Sort: (1, 0)

Next:

- Remove 1, subtract 1 from next 1 element:

Remaining sequence: (0)

Subtract 1 from 0: since 0 is less than 1, the process cannot proceed, indicating the sequence is potentially graphical.

Observation: The sequence process terminates successfully, with all zeros, indicating the sequence is graphical.

Step 4: Confirm with Erdős–Gallai Inequalities

The Erdős–Gallai theorem states that a non-increasing sequence of non-negative integers $d_1 \geq d_2 \geq ... \geq d_n$ is graphical if and only if:

$$
\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k), \quad \text{for all } 1 \leq k \leq n,
$$

and the total sum is even.

Let's verify for some values of k:

- $k=1:$
Sum $= 5$
RHS $= 1 \times 0 + \sum_{i=2}^{7} \min(d_i, 1) = 0 + (\min(4,1)+\min(3,1)+\min(3,1)+\min(2,1)+\min(2,1)+\min(1,1))$
$= 0 + (1+1+1+1+1+1) = 6$
$5 \leq 6$ — OK.

- $k=2:$
Sum $= 5+4=9$
RHS $= 2 \times 1 + \sum_{i=3}^{7} \min(d_i, 2) = 2 + (\min(3,2)+\min(3,2)+\min(2,2)+\min(2,2)+\min(1,2))$
$= 2 + (2+2+2+2+1) = 2 + 9 = 11$
$9 \leq 11$ — OK.

- $k=3:$
Sum $= 5+4+3=12$
RHS $= 3 \times 2 + \sum_{i=4}^{7} \min(d_i, 3) = 6 + (\min(3,3)+\min(2,3)+\min(2,3)+\min(1,3))$
$= 6 + (3+2+2+1) = 6 + 8 = 14$
$12 \leq 14$ — OK.

- $k=4:$
Sum $= 5+4+3+3=15$
RHS $= 4 \times 3 + \sum_{i=5}^{7} \min(d_i, 4) = 12 + (\min(2,4)+\min(2,4)+\min(1,4))$
$= 12 + (2+2+1) = 12 + 5 = 17$
$15 \leq 17$ — OK.

- $k=5:$
Sum $= 20$
RHS $= 5 \times 4 + \sum_{i=6}^{7} \min(d_i, 5) = 20 + (\min(2,5)+\min(1,5)) = 20 + (2+1) = 23$
$20 \leq 23$ — OK.

All inequalities are satisfied. The total sum is even (20). Therefore, the sequence is graphical according to the Erdős–Gallai theorem.

Constructing the Graph

Since the sequence is graphical, we can construct a simple graph with the given degree sequence. Let's sketch a possible construction:

1. Vertices:

Label vertices as \(V_1, V_2, V_3, V_4, V_5, V_6, V_7\).

2. Assign degrees:

- \(V_1\): degree 5
- \(V_2\): degree 4
- \(V_3, V_4\): degree 3
- \(V_5, V_

Frequently Asked Questions

Is the degree sequence 5, 4, 3, 3, 2, 2, 1 realizable as a simple graph?

Yes, the sequence is graphical, meaning there exists a simple graph with this degree sequence.

Can the Havel-Hakimi algorithm be used to verify if 5, 4, 3, 3, 2, 2, 1 is a graphical sequence?

Yes, applying the Havel-Hakimi algorithm confirms whether the sequence is graphical; in this case, it is.

Is the degree sequence 5, 4, 3, 3, 2, 2, 1 a valid degree sequence for a simple graph?

True, since the sequence passes the graphicality tests such as Havel-Hakimi, it is valid.

Does the sum of the degrees in the sequence 5, 4, 3, 3, 2, 2, 1 indicate a possible simple graph?

Yes, the sum of degrees is 20, which is even, satisfying a necessary condition for graphical sequences.

Is it possible to construct a simple graph with the degree sequence 5, 4, 3, 3, 2, 2, 1?

Yes, constructive methods confirm that such a graph can be built.

Does the degree sequence 5, 4, 3, 3, 2, 2, 1 violate any graphicality conditions?

No, it does not violate any conditions; it satisfies the basic requirements for being graphical.

Is the statement 'There can be a graph with degree sequence 5, 4, 3, 3, 2, 2, 1' true?

True, as the sequence is graphical and such a graph can exist.

Additional Resources

There Can Be A Graph With Degree Sequence 5, 4, 3, 3, 2, 2, 1 T/F?
Determining whether a specific degree sequence can be realized by a simple graph is a classic problem in graph theory. The question "There Can Be A Graph With Degree Sequence 5, 4, 3, 3, 2, 2, 1 T/F?" challenges us to analyze whether such a degree sequence is graphic—that is, whether a simple, undirected graph exists whose vertices have degrees exactly matching this sequence. In this article, we'll explore the principles, techniques, and step-by-step methods to answer this question thoroughly, providing a comprehensive guide to understanding degree sequences and their realizability.

Understanding Degree Sequences and Their Significance

A degree sequence of a graph is a list of the degrees of its vertices, typically arranged in non-increasing order. For example, in the sequence 5, 4, 3, 3, 2, 2, 1, the vertices have degrees 5, 4, 3, 3, 2, 2, and 1 respectively.

Why is this important?
Graph theorists often seek to determine whether a given sequence is graphic—meaning it corresponds to at least one simple graph. This problem has applications in network theory, social sciences, biology, and computer science, where understanding feasible degree distributions is crucial.

The Core Question: Is the Sequence Graphic?

Given the sequence:
5, 4, 3, 3, 2, 2, 1

Is there a simple graph with these vertex degrees?
The answer isn't immediately clear; it requires applying known theorems and algorithms. The main tools used include:

- Havel-Hakimi Algorithm
- Erdős–Gallai Theorem

- Degree sum considerations

Step 1: Basic Validity Checks

Before diving into complex algorithms, perform quick checks:

1. Sum of degrees must be even.
The Handshaking Lemma states that the sum of degrees in any graph must be even because each edge contributes 2 to the total degree sum.

2. Degrees are all less than or equal to the number of vertices minus one.
Each vertex can't connect to more vertices than actually exist.

Let's verify these:

- Sum of degrees:
5 + 4 + 3 + 3 + 2 + 2 + 1 = 20

- Is 20 even? Yes.
Pass.

- Number of vertices: 7
- Max degree should be ≤ 6 (since each vertex can connect to all other 6 vertices).
Max degree is 5, which is ≤ 6.
Pass.

Step 2: Applying the Havel-Hakimi Algorithm

The Havel-Hakimi process is a constructive method to determine if a degree sequence is graphic:

Procedure:

1. Order the sequence in non-increasing order.
Already ordered: 5, 4, 3, 3, 2, 2, 1

2. Remove the first degree (highest), say d.
- d = 5

3. Reduce the next d degrees by 1 each, since the highest-degree vertex connects to d other vertices.
- The next 5 degrees are: 4, 3, 3, 2, 2

4. Update the sequence:

- Subtract 1 from each of these five degrees:
4 → 3
3 → 2
3 → 2

2 → 1
2 → 1

- The remaining degrees: 1 (original), and the unmodified degrees beyond those processed.

- The sequence now:
Remaining degrees: 3, 2, 2, 1, 1, 1

5. Rearranged in non-increasing order:
3, 2, 2, 1, 1, 1

6. Repeat the process with the new sequence.

Step 3: Iterative Application

Let's go through the steps:

- Sequence: 3, 2, 2, 1, 1, 1

- Highest degree: 3

- Remove 3, reduce the next 3 degrees by 1: 2, 2, 1

- Updated degrees:

- 2 → 1
- 2 → 1
- 1 → 0

- Remaining degrees: 1, 1, 0, 1, 1

- Sort in non-increasing order: 1, 1, 1, 1, 0

Next iteration:

- Highest degree: 1

- Remove 1; reduce next 1 degree by 1:

- 1 → 0

- Remaining sequence: 1, 1, 0, 0

Sort: 1, 1, 0, 0

Next iteration:

- Highest degree: 1

- Remove 1; reduce next 1 degree by 1:

- 1 → 0

- Remaining: 0, 0

Sequence is now: 0, 0

All zeros, which indicates the process terminates successfully.

Conclusion: The sequence 5, 4, 3, 3, 2, 2, 1 is graphic.

Summary of the Havel-Hakimi Process:

- The sequence reduces to all zeros without contradiction.
- This confirms that a simple graph with degree sequence 5, 4, 3, 3, 2, 2, 1 exists.

Additional Validation: The Erdős–Gallai Theorem

While the Havel-Hakimi algorithm is sufficient for practical purposes, the Erdős–Gallai conditions provide a more theoretical approach.

The theorem states that a sequence $d_1 \geq d_2 \geq \dots \geq d_n$ is graphic if and only if:

1. $\sum_{i=1}^k d_i \leq k(k - 1) + \sum_{i=k+1}^n \min(d_i, k)$ for all $1 \leq k \leq n$

2. $\sum_{i=1}^n d_i$ is even (already verified)

Applying this to our sequence:

- For various k, verify the inequality.

For instance, for $k=1$:

- LHS: 5
- RHS: $1 \times (1 - 1) + \sum_{i=2}^7 \min(d_i, 1)$
= 0 + (min(4,1)+min(3,1)+min(3,1)+min(2,1)+min(2,1)+min(1,1))
= 0 + (1+1+1+1+1+1) = 6

Check: 5 ≤ 6 → True.

Similarly, for $k=2$:

- LHS: 5 + 4 = 9
- RHS: $2 \times (2 - 1) + \sum_{i=3}^7 \min(d_i, 2)$
= 2 + (min(3,2)+min(3,2)+min(2,2)+min(2,2)+min(1,2))
= 2 + (2+2+2+2+1) = 2 + 9 = 11

Check: 9 ≤ 11 → True.

Continuing for other \(k\) confirms the sequence is graphic.

Final Determination: True — There Can Be A Graph With Degree Sequence 5, 4, 3, 3, 2, 2, 1

The comprehensive analysis, employing the Havel-Hakimi algorithm and the Erdős–Gallai theorem, confirms that the degree sequence in question is realizable.

Practical Implications and Additional Considerations

Why is understanding whether a degree sequence is graphic important?
In designing networks, social graphs, or biological systems, constraints on degrees often arise naturally. The ability to verify realizability helps in:

- Modeling real-world networks with specified properties.
- Generating graphs with particular degree distributions for simulations.
- Understanding the structure and limitations inherent in certain degree distributions.

Limitations and caveats:

- The methods discussed assume simple graphs (no loops, no multiple edges).
- For directed graphs or multigraphs, different criteria apply.
- Large sequences may require computational algorithms for verification.

Summary and Key Takeaways

- The degree sequence 5, 4, 3, 3, 2, 2, 1 passes basic validity tests.
- The Havel-Hakimi algorithm confirms its realizability through iterative reduction.
- The Erdős–Gallai theorem provides a theoretical guarantee.
- Therefore, the statement "There Can Be A Graph With Degree Sequence 5, 4, 3,

[There Can Be A Graph With Degree Sequence 5 4 3 3 2 2 1 T F]

There Can Be A Graph With Degree Sequence 5 4 3 3 2 2 1 T F

Related Articles

- Hildegard's Music Is Similar To Gregorian Chant In That It
- Select The Most Stable Conformer Of Cis-cyclohexane-1 3-diol
- Consider A Cdf For A Random Variable X Given By Fx(x)=1-1/x 1