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In the realm of differential equations, solving systems of first-order linear differential equations is a fundamental skill with applications across engineering, physics, and applied mathematics. The problem at hand involves solving the system Y' = AY, where A is a specified 2x2 matrix, and an initial condition is provided. This article will guide you through a step-by-step process to find the explicit solution for the vector function Y(t), starting from the matrix A, determining eigenvalues and eigenvectors, and constructing the general solution.

Understanding the Problem

What is the Differential Equation System?

The problem involves solving a system of two first-order linear differential equations represented as:

Y' = AY

where:

- Y(t) is a vector function: Y(t) = [y_1(t), y_2(t)]^T
- A is a 2x2 matrix given as:

A = | -6 24 |
| 1 -1 |

- The initial condition is Y(1) = [1, ?], but as per the problem statement, assume the initial vector is Y(1) = [1, 0]^T (or adjust as necessary based on the context).

Why is solving Y' = AY important?

Understanding and solving such systems allows us to model complex phenomena like population dynamics, electrical circuits, and mechanical systems. Finding explicit solutions helps predict behavior over time and analyze stability.

Step-by-Step Solution Approach

Solving the system involves several key steps:

1. Find the eigenvalues of matrix A.
2. Find corresponding eigenvectors.
3. Construct the general solution using eigenvalues and eigenvectors.
4. Apply initial conditions to find particular solutions.

Step 1: Find Eigenvalues of A

Characteristic Equation

Eigenvalues λ are solutions to:

$\det(A - \lambda I) = 0$

where I is the identity matrix.

For matrix A:

A - λI = | -6 - λ 24 |
| 1 -1 - λ |

Calculate the determinant:

$(-6 - \lambda)(-1 - \lambda) - (24)(1) = 0$

Expand:

$(-6 - \lambda)(-1 - \lambda) - 24 = 0$

Multiply out:

$(-6)(-1) + (-6)(-\lambda) + (-\lambda)(-1) + (-\lambda)(-\lambda) - 24 = 0$

Simplify:

$6 + 6\lambda + \lambda + \lambda^2 - 24 = 0$

Combine like terms:

$\lambda^2 + (6\lambda + \lambda) + (6 - 24) = 0$

$\lambda^2 + 7\lambda - 18 = 0$

Eigenvalues Calculation

Solve quadratic:

$\lambda^2 + 7\lambda - 18 = 0$

Using quadratic formula:

$\lambda = [-7 \pm \sqrt{(49 - 4(1)(-18))}]/(2)$

Calculate discriminant:

$49 + 72 = 121$

Square root:

$\sqrt{121} = 11$

Eigenvalues:

$\lambda_1 = [-7 + 11]/2 = 4/2 = 2$

$\lambda_2 = [-7 - 11]/2 = -18/2 = -9$

Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -9$

Step 2: Find Eigenvectors

Eigenvector corresponding to λ = 2

Solve:

$(A - 2I)v = 0$

A - 2I:

| -6 - 2 24 | = | -8 24 |
| 1 -1 - 2 | | 1 -3 |

Set up the system:

$-8x + 24y = 0$ ...(1)

$x - 3y = 0$ ...(2)

From (2): $x = 3y$

Plug into (1):

$-8(3y) + 24y = 0$

$-24y + 24y = 0$

$0 = 0$ (consistent)

Choose $y = 1$, then $x = 3$

Eigenvector:

$v_1 = [3, 1]^T$

Eigenvector corresponding to $\lambda = -9$

Solve:

$(A + 9I)v = 0$

$A + 9I$:

$$\begin{vmatrix} -6 + 9 & 24 \\ 1 & -1 + 9 \end{vmatrix} = \begin{vmatrix} 3 & 24 \\ 1 & 8 \end{vmatrix}$$

Set up the system:

$3x + 24y = 0$ ...(1)

$x + 8y = 0$ ...(2)

From (2): $x = -8y$

Substitute into (1):

$3(-8y) + 24y = 0$

$-24y + 24y = 0$

Again, consistent.

Choose $y = 1$, then $x = -8$

Eigenvector:

$v_2 = [-8, 1]^T$

Step 3: Construct the General Solution

The general solution for the system is:

$Y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Plugging in the eigenvalues and eigenvectors:

$Y(t) = c_1 e^{2 t} [3, 1]^T + c_2 e^{-9 t} [-8, 1]^T$

Expressed component-wise:

$y_1(t) = 3 c_1 e^{2 t} - 8 c_2 e^{-9 t}$

$y_2(t) = c_1 e^{2 t} + c_2 e^{-9 t}$

Step 4: Apply Initial Conditions

Assuming the initial condition is $Y(1) = [1, 0]^T$, substitute $t = 1$:

$y_1(1) = 3 c_1 e^{2} - 8 c_2 e^{-9} = 1$

$y_2(1) = c_1 e^{2} + c_2 e^{-9} = 0$

Set up the system:

1) $3 c_1 e^{2} - 8 c_2 e^{-9} = 1$

2) $c_1 e^{2} + c_2 e^{-9} = 0$

Solve for c_1 and c_2:

From (2):

$c_1 e^{2} = - c_2 e^{-9}$

$c_1 = - c_2 e^{-9} / e^{2} = - c_2 e^{-11}$

Substitute into (1):

$3 (- c_2 e^{-11}) e^{2} - 8 c_2 e^{-9} = 1$

Simplify:

$-3 c_2 e^{-11} e^{2} - 8 c_2 e^{-9} = 1$

$e^{-11} e^{2} = e^{-9}$

Thus:

$-3 c_2 e^{-9} - 8 c_2 e^{-9} = 1$

Combine:

$(-3 - 8) c_2 e^{-9} = 1$

$-11 c_2 e^{-9} = 1$

Solve for c_2:

$c_2 = -1 / (11 e^{-9}) = - e^{9} / 11$

Now, find c_1:

$c_1 = - c_2 e^{-11} / e^{2} = - (- e^{9} / 11) e^{-11} / e^{2}$

Simplify numerator:

$(e^{9} / 11) e^{-11} = e^{9 - 11} / 11 = e^{-2} / 11$

Divide by e^{2}:

$c_1 = - (e^{-2} / 11) / e^{2} = - (e^{-2} / 11) e^{-2} = - e^{-4} / 11$

Final Solution

Putting it all together, the particular solution satisfying the initial condition is:

$y_1(t) = 3 c_1 e^{2 t} - 8 c_2 e^{-9 t} = 3 \left(- \frac{e^{-4}}{11}\right) e^{2 t} - 8 \left(- \frac{e^{9}}{11}\right) e^{-9 t}$

Simplify:

$y_1(t) = - \frac{3 e^{-4}}{11} e^{2 t} + \frac{8 e^{9}}{11} e^{-9 t}$

Similarly,

$y_2(t) = c_1 e^{2 t} + c_2 e^{-9 t} = - \frac{e^{-4}}{11} e^{2 t} - \frac{e^{9}}{11} e^{-9 t}$

Therefore, the explicit solution is:

\[Y(t) = \left[y_1(t), y_2(t) \right]^T = \left[- \frac{3 e^{-4}}{11} e^{2 t} + \frac

Frequently Asked Questions

What type of differential equation is 8 2 Solve Y' = AY, and what does it represent?

It is a system of first-order linear differential equations represented in matrix form, modeling how the vector function Y evolves over time under the influence of matrix A.

Given the matrix A and initial condition Y(1) = [1], how can we find the solution Y(t)?

We can find the solution using matrix exponential methods: Y(t) = e^{A(t-1)} Y(1), where e^{A(t-1)} is the matrix exponential of A scaled by (t-1).

What are the eigenvalues and eigenvectors of matrix A, and how do they help solve the differential equation?

Eigenvalues and eigenvectors of A determine the behavior of the system; by diagonalizing A or using Jordan forms, we can compute e^{A(t-1)} to find explicit solutions for Y(t).

How does the initial condition Y(1) = [1] influence the solution of the differential equation?

The initial condition provides the starting value at t=1, allowing us to determine the specific solution among the family of solutions derived from the matrix exponential.

What challenges might arise when computing e^{A(t-1)} for the given matrix A, and how can they be addressed?

Challenges include eigenvalue computation and matrix diagonalization; these can be addressed using numerical methods or Jordan normal form to compute the matrix exponential accurately.

Additional Resources

8 2 Solve Y' = AY, Where -6 24 (1) A= -1 8 4 2 -12 -6 And Y(1) = [1]

Introduction

The study of systems of differential equations is a cornerstone of mathematical analysis, with applications spanning physics, engineering, economics, and beyond. One fundamental class of such systems is the linear system of the form:

\[Y' = A Y \]

where Y is a vector-valued function and A is a constant matrix. The solution process often involves finding eigenvalues and eigenvectors of A, enabling the explicit construction of the general solution. This article delves into a specific instance of such a system, characterized by a matrix A and an initial condition:

$$
A = \begin{bmatrix}
-6 & 24 \\
-1 & 8
\end{bmatrix}
,\quad
Y(1) = \begin{bmatrix} 1 \end{bmatrix}
$$

The goal is to thoroughly analyze, solve, and interpret the system, providing an instructive guide for researchers and students alike.

Context and Significance of the Problem

Understanding the solution to $Y' = AY$ hinges on the properties of the matrix A. The eigenvalues and eigenvectors determine the system's stability and dynamics. This specific matrix A exhibits characteristics that require a careful spectral analysis, including the calculation of eigenvalues, eigenvectors, and the matrix exponential.

The initial condition $Y(1) = [1]$ adds an initial value to the problem, enabling the determination of particular solutions. Such initial value problems are common in modeling real-world phenomena where the state at a specific time is known, and the evolution thereafter is to be determined.

Detailed Analysis of the System

1. Restating the System

Given:

$$
Y' = A Y, \quad \text{where} \quad
A = \begin{bmatrix}
-6 & 24 \\
-1 & 8
\end{bmatrix}
,\quad
Y(1) = \begin{bmatrix} 1 \end{bmatrix}
$$

It appears that the initial condition as provided is a single scalar. Since the system is two-dimensional (implied by the matrix A), the initial condition should likely be a vector, for example, $Y(1) = \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix}$. For the purposes of this analysis, we will assume the initial condition is:

$$
Y(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

or, if the problem intends a scalar, perhaps the initial data is only for the first component, and the second component is unspecified or zero. This assumption is necessary for a meaningful solution.

---

2. Eigenvalue and Eigenvector Calculation

Step 1: Find the eigenvalues λ of A by solving the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

$$
\det \begin{bmatrix}
-6 - \lambda & 24 \\
-1 & 8 - \lambda
\end{bmatrix} = 0
$$

Calculating the determinant:

$$
(-6 - \lambda)(8 - \lambda) - (-1)(24) = 0
$$

$$
(-6 - \lambda)(8 - \lambda) + 24 = 0
$$

Expanding:

$$
(-6)(8 - \lambda) - \lambda(8 - \lambda) + 24 = 0
$$

$$
-48 + 6 \lambda - 8 \lambda + \lambda^2 + 24 = 0
$$

Simplify:

$$
\lambda^2 - 2 \lambda - 24 = 0
$$

Step 2: Solve the quadratic

$$
\lambda^2 - 2 \lambda - 24 = 0
$$

$$
\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-24)}}{2} = \frac{2 \pm \sqrt{4 + 96}}{2} = \frac{2 \pm \sqrt{100}}{2}
$$

$$
\lambda = \frac{2 \pm 10}{2}
$$

Thus,

$$
\boxed{
\lambda_1 = 6, \quad \lambda_2 = -4
}
$$

---

3. Eigenvector Computation

Eigenvector for $\lambda_1 = 6$:

Solve $(A - 6I) \mathbf{v} = 0$:

$$
A - 6I = \begin{bmatrix}
-6 - 6 & 24 \\
-1 & 8 - 6
\end{bmatrix} = \begin{bmatrix}
-12 & 24 \\
-1 & 2
\end{bmatrix}
$$

Set up the equations:

$$
-12 v_1 + 24 v_2 = 0
$$
$$
-1 v_1 + 2 v_2 = 0
$$

From the second:

$$
- v_1 + 2 v_2 = 0 \Rightarrow v_1 = 2 v_2
$$

Substitute into the first:

$$
-12 (2 v_2) + 24 v_2 = -24 v_2 + 24 v_2 = 0
$$

which holds for any v_2. Choose $v_2 = 1$:

$$
v^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}
$$

Eigenvector for $\lambda_2 = -4$:

Calculate $A + 4I$:

$$
A + 4I = \begin{bmatrix}
-6 + 4 & 24 \\
-1 & 8 + 4
\end{bmatrix} = \begin{bmatrix}
-2 & 24 \\
-1 & 12
\end{bmatrix}
$$

Set up the equations:

$$
-2 v_1 + 24 v_2 = 0
$$
$$
-1 v_1 + 12 v_2 = 0
$$

From the second:

$$
- v_1 + 12 v_2 = 0 \Rightarrow v_1 = 12 v_2
$$

Plug into the first:

$$
-2 (12 v_2) + 24 v_2 = -24 v_2 + 24 v_2 = 0
$$

which is valid. Choose $v_2 = 1$:

$$
v^{(2)} = \begin{bmatrix} 12 \\ 1 \end{bmatrix}
$$

Constructing the General Solution

With eigenvalues and eigenvectors, the general solution for the system is:

$$
Y(t) = c_1 e^{\lambda_1 t} v^{(1)} + c_2 e^{\lambda_2 t} v^{(2)}
$$

or explicitly:

$$
Y(t) = c_1 e^{6 t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-4 t} \begin{bmatrix} 12 \\ 1 \end{bmatrix}
$$

Application of Initial Conditions and Particular Solution

Assuming initial condition:

$$
Y(1) = \begin{bmatrix} y_1(1) \\ y_2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

we substitute $t=1$:

$$
Y(1) = c_1 e^{6} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-4} \begin{bmatrix} 12 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

which leads to the system:

$$
2 c_1 e^6 + 12 c_2 e^{-4} = 1
$$
$$
c_1 e^6 + c_2 e^{-4} = 0
$$

Express c_1 from the second:

$$
c_1 = - c_2 e^{-4} / e^6 = - c_2 e^{-10}
$$

Substitute into the first:

$$
2 (- c_2 e^{-10}) e^{6} + 12 c_2 e^{-4} = 1
$$

Simplify:

$$
-2 c_2 e^{-10 + 6} + 12 c_2 e^{-4} = 1
$$

$$
-2 c_2 e^{-4} + 12 c_2 e^{-4} = 1
$$

$$
(-2 + 12) c_2 e^{-4} = 1
$$

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