Calculate The Double Integral. 6x/(1 + Xy) DA, R = [0, 6] X [0, 1]

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Understanding how to evaluate double integrals is fundamental in multivariable calculus, especially when analyzing functions over specific regions. In this article, we will explore the process of calculating the double integral of the function $\frac{6x}{1 + xy}$ over the rectangular region $R = [0, 6] \times [0, 1]$. We will break down each step meticulously to ensure clarity and comprehension, making it accessible for students and enthusiasts alike.

Introduction to Double Integrals

Double integrals are a powerful tool in calculus used to compute the volume under a surface or the accumulation of a quantity over a two-dimensional region. They generalize the concept of single-variable integrals to two variables, providing insights into functions of two variables.

Key Concepts:
- The double integral of a function $f(x, y)$ over a region R is denoted as $\iint_R f(x, y) \, dA$.
- The region R can be rectangular, circular, or more complex shapes, and the limits of integration depend on this region.
- The order of integration (integrating with respect to x first or y first) can sometimes simplify calculations.

Understanding the Region of Integration

The region R over which we are integrating is given as:
$$
R = [0, 6] \times [0, 1]
$$
This defines a rectangle in the xy-plane:
- x ranges from 0 to 6.
- y ranges from 0 to 1.

This rectangular region simplifies the process as the limits are constant and independent, making it straightforward to set up the double integral.

Setting Up the Double Integral

The double integral for the given function over R can be written as:

$$
\iint_R \frac{6x}{1 + xy} \, dA
$$

Since the region is rectangular with constant bounds, the integral can be expressed as an iterated integral in two ways:

Option 1: Integrate with respect to y first, then x:

$$
\int_{x=0}^{6} \int_{y=0}^{1} \frac{6x}{1 + xy} \, dy \, dx
$$

Option 2: Integrate with respect to x first, then y:

$$
\int_{y=0}^{1} \int_{x=0}^{6} \frac{6x}{1 + xy} \, dx \, dy
$$

Choosing the order depends on which integral is easier to evaluate. Given the structure of the function, integrating with respect to y first appears more manageable because the denominator contains xy.

Evaluating the Inner Integral

Let's proceed with the order:

$$
\int_{x=0}^{6} \left(\int_{y=0}^{1} \frac{6x}{1 + xy} \, dy \right) dx
$$

Focus on the inner integral:

$$
I(x) = \int_{0}^{1} \frac{6x}{1 + xy} \, dy
$$

Calculating $I(x)$

Notice that $6x$ is a constant with respect to y, so it can be factored out:

$$
I(x) = 6x \int_0^1 \frac{1}{1 + xy} \, dy
$$

Now, consider the integral:

$$
J = \int_0^1 \frac{1}{1 + xy} \, dy
$$

This is a standard integral, which can be solved via substitution.

Solving the Inner Integral J

Substitution:

Let:

$$
u = 1 + xy
$$

then:

$$
du = x \, dy \Rightarrow dy = \frac{du}{x}
$$

Change of limits:

When $y = 0$:

$$
u = 1 + x \cdot 0 = 1
$$

When $y = 1$:

$$
u = 1 + x \cdot 1 = 1 + x
$$

Rewriting the integral \(J \):

\[
J = \int_{u=1}^{1 + x} \frac{1}{u} \cdot \frac{du}{x} = \frac{1}{x} \int_1^{1 + x} \frac{1}{u} \,
du
\]

Evaluating the integral:

\[
J = \frac{1}{x} \left[\ln |u| \right]_1^{1 + x} = \frac{1}{x} \left(\ln(1 + x) - \ln 1 \right)
\]

Since \(\ln 1 = 0 \):

\[
J = \frac{1}{x} \ln(1 + x)
\]

Recall that:

\[
I(x) = 6x \times J = 6x \times \frac{1}{x} \ln(1 + x) = 6 \ln(1 + x)
\]

Thus, the inner integral simplifies to:

\[
I(x) = 6 \ln(1 + x)
\]

Evaluating the Outer Integral

Now, we substitute \(I(x) \) back into the outer integral:

\[
\int_0^6 I(x) \, dx = \int_0^6 6 \ln(1 + x) \, dx
\]

This simplifies to:

\[
6 \int_0^6 \ln(1 + x) \, dx
\]

Calculating $\int_0^6 \ln(1 + x) \, dx$

Method: Integration by parts

Let:

$$
u = \ln(1 + x) \Rightarrow du = \frac{1}{1 + x} dx
$$
$$
dv = dx \Rightarrow v = x
$$

Applying integration by parts:

$$
\int u \, dv = uv - \int v \, du
$$

Thus:

$$
\int_0^6 \ln(1 + x) \, dx = \left[x \ln(1 + x) \right]_0^6 - \int_0^6 x \cdot \frac{1}{1 + x} dx
$$

Evaluating the Boundary Term

At $x = 6$:

$$
6 \ln(1 + 6) = 6 \ln 7
$$

At $x = 0$:

$$
0 \times \ln(1 + 0) = 0
$$

So, the boundary term is:

$$
6 \ln 7 - 0 = 6 \ln 7
$$

Evaluating $\int_0^6 \frac{x}{1 + x} dx$

Simplify the integrand:

$$
\frac{x}{1 + x} = \frac{(1 + x) - 1}{1 + x} = 1 - \frac{1}{1 + x}
$$

Therefore:

$$
\int_0^6 \frac{x}{1 + x} dx = \int_0^6 \left(1 - \frac{1}{1 + x}\right) dx = \int_0^6 1 \, dx - \int_0^6 \frac{1}{1 + x} dx
$$

Calculate each:

- $\int_0^6 1 \, dx = 6$
- $\int_0^6 \frac{1}{1 + x} dx = \left[\ln |1 + x| \right]_0^6 = \ln 7 - \ln 1 = \ln 7$

Putting it together:

$$
\int_0^6 \frac{x}{1 + x} dx = 6 - \ln 7
$$

Final Expression for the Integral

Recall:

$$
\int_0^6 \ln(1 + x) dx = 6 \ln 7 - \left(6 - \ln 7 \right) = 6 \ln 7 - 6 + \ln 7
$$

Simplify:

$$
= (6 \ln 7 + \ln 7) - 6 = 7 \ln 7 - 6
$$

Now, multiply by the factor outside the integral:

$$
6 \times (7 \ln 7 - 6) = 6 \times 7 \ln 7 - 6 \times 6 = 42 \ln 7 - 36
$$

Final Result of the

Frequently Asked Questions

How do I set up the double integral for evaluating the function 6x / (1 + xy) over the region R = [0,6] x [0,1]?

You set up the double integral as $\int_0^1 \int_0^6 (6x)/(1 + xy)\, dx\, dy$, integrating with respect to x first from 0 to 6, then y from 0 to 1.

What is the best method to evaluate the inner integral $\int_0^6 (6x)/(1 + xy)\, dx$ for a fixed y?

You can attempt substitution, such as u = 1 + xy, which simplifies the integral. Remember to express dx in terms of du and adjust limits accordingly.

Can changing the order of integration simplify evaluating $\int_0^1 \int_0^6 (6x)/(1 + xy)\, dx\, dy$?

In this case, since the region is a rectangle, changing the order doesn't simplify much. Focus on substitution within the inner integral for efficiency.

How does the substitution u = 1 + xy help in evaluating the double integral?

It transforms the integrand into a form involving du, often simplifying the integral into a rational function that can be integrated directly.

What are potential challenges when integrating 6x / (1 + xy) over the given region?

The main challenge is handling the dependence on both x and y in the denominator, which can be addressed via substitution or recognizing standard integral forms.

Once the inner integral is evaluated, how do I proceed to compute the outer integral over y?

After finding the antiderivative of the inner integral with respect to x, substitute the limits and then integrate the resulting expression with respect to y from 0 to 1.

Are there any special techniques or formulas that can help evaluate the double integral of 6x / (1 + xy)?

Yes, substitution techniques such as u = 1 + xy, or recognizing the integral as a derivative of a

logarithmic function, can simplify the process.

Additional Resources

Calculating the Double Integral of $\frac{6x}{1 + xy}$ over the Region $D = [0,6] \times [0,1]$

Introduction

In the realm of multivariable calculus, double integrals serve as essential tools for determining the volume under a surface, average values over regions, and for solving complex real-world problems involving multiple variables. Today, we delve into a specific and intriguing double integral:

$$
\iint_D \frac{6x}{1 + xy} \, dA
$$

where $D = [0, 6] \times [0, 1]$.

This integral not only exemplifies the application of double integration but also showcases the strategic choice of variable substitution to simplify the computation. Our goal is to explore this integral thoroughly, from understanding the region to performing the integration step-by-step, all while providing insights into the techniques involved.

Understanding the Problem: The Integral and Region

The Region D

The region of integration, D, is a rectangular area in the xy-plane:

- x varies from 0 to 6
- y varies from 0 to 1

This straightforward rectangular domain simplifies the process, allowing us to focus on the integrand's behavior and the substitution strategy.

The Integrand: $\frac{6x}{1 + xy}$

The integrand involves a rational function with a product xy in the denominator, hinting at potential substitution methods to simplify the expression. Notably, the presence of xy suggests that a substitution involving $u = xy$ could be effective.

Strategizing the Integration Approach

Choice of Method

Given the structure of the integrand, the most promising approach involves:

- Variable substitution to simplify the integrand
- Iterated integration, integrating with respect to one variable first and then the other

Why Substitution?

The term $(1 + xy)$ in the denominator complicates direct integration with respect to x or y. However, if we can transform the integrand into a form where the denominator becomes a simple function of the new variable, the integral becomes more manageable.

Potential Substitutions

- $u = xy$: Since u combines both variables, it suggests that integrating with respect to x (or y) while considering this substitution could simplify the process.

- Alternatively, fixing y and integrating with respect to x might be effective, as x appears explicitly in the numerator and denominator.

Step-by-Step Solution

Step 1: Express the Double Integral

The double integral over the region D can be written as an iterated integral:

$$
\int_{y=0}^{1} \int_{x=0}^{6} \frac{6x}{1 + xy} \, dx \, dy
$$

or vice versa. To optimize the process, we'll integrate with respect to x first, then y.

Step 2: Inner Integral with respect to x