Find All The Second Partial Derivatives. Z = Arctan X + Y 1 Xy

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Understanding how to compute second-order partial derivatives is fundamental in multivariable calculus, especially when analyzing the curvature, concavity, and other properties of multivariable functions. In this article, we will explore the process of finding all second partial derivatives for the function:

$$ Z = \arctan X + Y \cdot \frac{1}{X Y} $$

or, more precisely, the function:

$$ Z = \arctan X + \frac{Y}{X Y} $$

which simplifies further as we analyze it.

Understanding the Function $Z = \arctan X + \frac{Y}{X Y}$

Before diving into the derivatives, it's essential to clearly understand the structure and components of the function.

Simplification of the Function

Let's first analyze the second term:

$$ \frac{Y}{X Y} $$

Provided $Y \neq 0$, this simplifies to:

$$ \frac{Y}{X Y} = \frac{1}{X} $$

Therefore, the function simplifies to:

$$ Z = \arctan X + \frac{1}{X} $$

This clarification is crucial because it reduces the complexity of the derivatives and allows us to focus on a function of X and Y, with Z expressed as:

$$ Z = \arctan X + \frac{1}{X} $$

which depends solely on X. However, since the original problem mentions $Z = \arctan X + Y \cdot \frac{1}{X Y}$, and assuming the original notation meant $Z = \arctan X + \frac{Y}{X Y}$, the simplification leads us to a function independent of Y. But if the initial expression was intended differently, clarify accordingly.

Clarification of the Function

If the original expression is:

$$ Z = \arctan X + Y \cdot \frac{1}{X Y} $$

then, as shown, it's equivalent to:

$$ Z = \arctan X + \frac{1}{X} $$

which is a function of X only.

In such a case, the derivatives with respect to Y would be zero, and second derivatives would reflect this.

Calculating the First Partial Derivatives

To find all second partial derivatives, we start with the first derivatives.

Partial Derivative of Z with respect to X

Given:

$$ Z = \arctan X + \frac{1}{X} $$

The derivative of $\arctan X$ with respect to X is:

$$ \frac{\partial}{\partial X} \arctan X = \frac{1}{1 + X^2} $$

The derivative of $\frac{1}{X}$ with respect to X is:

$$ \frac{\partial}{\partial X} \left(\frac{1}{X} \right) = -\frac{1}{X^2} $$

Thus, the first partial derivative of Z with respect to X is:

$$
\boxed{
Z_X = \frac{1}{1 + X^2} - \frac{1}{X^2}
}
$$

Partial Derivative of Z with respect to Y

Since Z does not explicitly depend on Y after simplification, its partial derivative with respect to Y is:

$$
Z_Y = 0
$$

Calculating the Second Partial Derivatives

Now, proceed to find all second derivatives, including mixed derivatives.

Second Partial Derivative with respect to X (Second Order)

Differentiate Z_X with respect to X:

$$
Z_{XX} = \frac{\partial}{\partial X} \left(\frac{1}{1 + X^2} - \frac{1}{X^2} \right)
$$

Compute separately:

1. Derivative of $\frac{1}{1 + X^2}$:

$$
\frac{\partial}{\partial X} \left(\frac{1}{1 + X^2} \right) = - \frac{2X}{(1 + X^2)^2}
$$

2. Derivative of $- \frac{1}{X^2}$:

$$
\frac{\partial}{\partial X} \left(- \frac{1}{X^2} \right) = - \left(- \frac{2}{X^3} \right) = \frac{2}{X^3}
$$

Adding these, the second derivative with respect to X is:

$$
Z_{XX} = - \frac{2X}{(1 + X^2)^2} + \frac{2}{X^3}
$$

Second Derivatives with Respect to Y

Since Z does not depend on Y, all derivatives involving Y are zero:

$$
Z_Y = 0
$$

$$
Z_{YY} = 0
$$

$$
Z_{XY} = Z_{YX} = 0
$$

Summary of All Second Partial Derivatives

Second Partial Derivative	Expression
Z_{XX}	$- \frac{2X}{(1 + X^2)^2} + \frac{2}{X^3}$
Z_{YY}	0
Z_{XY}	0
Z_{YX}	0

Implications and Applications

Concavity and Curvature Analysis

Knowing the second derivatives helps analyze the curvature of the surface described by Z. For example:

- If $Z_{XX} > 0$, the function is locally convex in the X-direction.
- If $Z_{XX} < 0$, it is locally concave in the X-direction.

Since $Z_{YY} = 0$, the surface is flat in the Y-direction, indicating no curvature along that axis.

Second Derivative Test for Multivariable Functions

The second derivatives are used to compute the Hessian determinant:

$$
H = Z_{XX} \cdot Z_{YY} - (Z_{XY})^2
$$

In this case:

$$
H = Z_{XX} \cdot 0 - 0^2 = 0
$$

A zero Hessian indicates a saddle point or a flat region, depending on the context.

Special Cases and Domain Considerations

Domain Restrictions

- The function $Z = \arctan X + \frac{1}{X}$ is defined for all $X \neq 0$.
- The derivatives involve terms like X^2 and X^3, which are well-defined except at $X=0$.

Behavior at Critical Points

- To find critical points, set $Z_X = 0$:

$$
\frac{1}{1 + X^2} - \frac{1}{X^2} = 0
$$

- Solve for X:

$$
\frac{1}{1 + X^2} = \frac{1}{X^2}
$$

$$
X^2 = 1 + X^2
$$

which simplifies to:

$$
0 = 1
$$

Thus, no real solutions, indicating no critical points for Z in X.

Conclusion

In this comprehensive analysis, we have:

- Simplified the original function to $Z = \arctan X + 1/X$.
- Calculated the first derivatives Z_X and Z_Y.
- Derived the second derivatives Z_{XX}, Z_{YY}, and Z_{XY}.
- Discussed their implications for the function's behavior, curvature, and critical points.

Understanding the second partial derivatives provides valuable insights into the surface's shape, stability, and concavity. This process exemplifies the importance of careful differentiation and interpretation in multivariable calculus.

Further Reading and Practice

To deepen your understanding of second derivatives in multivariable calculus, consider exploring:

- Hessian matrix and its applications.
- Second derivative test in multiple variables.
- Curvature and saddle points analysis.
- Practical problems involving second derivatives in physics and engineering.

Practice with similar functions, including those with more complex dependencies, to strengthen your skills in calculating and interpreting second partial derivatives.

Note: If the original function intended was different (for example, involving Y explicitly in the second term without simplification), please clarify the expression to adjust the derivatives accordingly.

Frequently Asked Questions

How do you find the second partial derivatives of Z

= arctan(x) + y / x?

First, compute the first partial derivatives $\partial Z/\partial x$ and $\partial Z/\partial y$, then differentiate these again with respect to x and y respectively to obtain the second partial derivatives, such as $\partial^2 Z/\partial x^2$, $\partial^2 Z/\partial y^2$, and $\partial^2 Z/\partial x \partial y$.

What is the value of the mixed second partial derivative ∂²Z/∂x∂y for Z = arctan(x) + y / x?

Since $Z = \arctan(x) + y / x$, the mixed partial derivative $\partial^2 Z/\partial x \partial y$ is found by differentiating $\partial Z/\partial y$ with respect to x, which results in $-y / x^2$.

Are the second partial derivatives of Z = arctan(x) + y / x continuous?

Yes, the second partial derivatives are continuous in the domain where $x \neq 0$, because arctan(x) and y / x are smooth functions for $x \neq 0$.

How do you verify the symmetry of mixed partial derivatives for Z = arctan(x) + y / x?

You compute both $\partial^2 Z/\partial x \partial y$ and $\partial^2 Z/\partial y \partial x$. If the function is sufficiently smooth, these mixed derivatives should be equal, which can be confirmed through explicit calculation.

What is the second partial derivative ∂²Z/∂x² for Z = arctan(x) + y / x?

The second partial derivative $\partial^2 Z/\partial x^2$ is given by the derivative of $\partial Z/\partial x = 1 / (1 + x^2) - y / x^2$, resulting in $-2x / (1 + x^2)^2 + 2y / x^3$.

How does the second partial derivative ∂²Z/∂y² look for the function Z = arctan(x) + y / x?

Since $\partial Z/\partial y = 1 / x$, differentiating again with respect to y yields zero, so $\partial^2 Z/\partial y^2 = 0$.

What is the significance of calculating second partial derivatives in this context?

Calculating second partial derivatives helps analyze the curvature, concavity, and possible points of inflection of the function Z, as well as verifying conditions like the Hessian determinant for classifying stationary points.

Can the second partial derivatives of Z = arctan(x) + y / x be used to determine concavity?

Yes, by examining the signs of the second derivatives and the Hessian matrix, you can determine the concavity or convexity of the function in different regions.

Additional Resources

Find All The Second Partial Derivatives. Z = Arctan X + Y / Xy – A Comprehensive Guide

When delving into the realm of multivariable calculus, one of the fundamental skills is calculating partial derivatives, particularly second partial derivatives. These derivatives provide critical insights into the behavior of functions of multiple variables, such as concavity, points of inflection, and the nature of stationary points. In this guide, we will explore how to find all the second partial derivatives of the function Z = Arctan X + Y / (Xy), breaking down each step meticulously to help you understand the process thoroughly.

Introduction

The function in question, Z = Arctan X + Y / (Xy), presents an interesting challenge due to its combination of inverse tangent and rational expressions involving multiple variables X and Y. Calculating second partial derivatives involves multiple steps: first finding the first partial derivatives with respect to each variable, then differentiating those results again with respect to the same or different variables.

Understanding how to compute these derivatives is essential not only for theoretical purposes but also for practical applications such as optimization, modeling, and analyzing the curvature of surfaces.

Understanding the Function

Let's analyze the function:

Z = Arctan X + Y / (Xy)

At first glance, this function is composed of:

- An inverse tangent function, Arctan X, which depends solely on X.
- A rational term, Y / (Xy), which involves both variables, and contains the product Xy in the denominator.

Note: The notation "Y / (Xy)" suggests the expression is Y divided by X times y, but to avoid ambiguity, it's common in mathematics to write it as Y / (X y). For clarity, we'll interpret the function as:

$$Z = \arctan(X) + (Y) / (X y)$$

where X and Y are variables, and y is the same as Y. To maintain consistency, we'll treat the variables as X and Y, and the function as:

$$Z = \arctan(X) + Y / (X Y)$$

which simplifies to:

$$Z = \arctan(X) + 1 / X$$

since $Y / (X Y) = 1 / X$, provided $Y \neq 0$.

However, if the original function was intended to be $Z = \arctan X + Y / (X y)$ with different variables, then clarification is needed. Assuming the variables are X and Y, and the function is:

$$Z = \arctan(X) + Y / (X Y) = \arctan(X) + 1 / X$$

then the function simplifies significantly, and the derivatives become straightforward.

Alternatively, if the function is written as:

$Z = \arctan(X) + (Y) / (X y)$, and Y and y are different variables, the calculation would be more involved.

Clarifying the Function: The Most Likely Interpretation

Given the notation, the most plausible interpretation is:

$$Z = \arctan X + Y / (X Y) = \arctan X + 1 / X$$

which simplifies the problem to derivatives of:

$$Z = \arctan X + 1 / X$$

OR

if the function is $Z = \arctan X + Y / (X y)$ with variables X and Y, and y is a constant or a different variable, then the derivatives with respect to X and Y would be different.

For completeness, let's proceed with the most general case:

General Case

$Z = \arctan X + (Y) / (X y)$

where:

- X and Y are variables.
- y may be a constant or another variable.

To ensure clarity, we'll treat $Z = \arctan X + Y / (X y)$, with y as a constant, simplifying the calculation. If y is a variable, the same process applies with respect to that variable.

Step-by-Step Calculation of Second Partial Derivatives

1. Compute the First Partial Derivatives

The process begins with calculating the first derivatives of Z with respect to each variable, X and Y.

Partial Derivative of Z with respect to X: Z_x

Given:

$$ Z = \arctan X + \frac{Y}{X y} $$

Assuming y is a constant:

$$
Z_x = \frac{d}{dX} \left(\arctan X \right) + \frac{d}{dX} \left(\frac{Y}{X y} \right)
$$

- Derivative of $\arctan X$:

$$
\frac{1}{1 + X^2}
$$

- Derivative of $\frac{Y}{X y}$:

Since Y and y are constants with respect to X, we treat them as constants:

$$
\frac{d}{dX} \left(\frac{Y}{X y} \right) = Y \cdot \frac{d}{dX} \left(\frac{1}{X y} \right) = Y \cdot \left(-\frac{1}{X^2 y} \right) = -\frac{Y}{X^2 y}
$$

\]

Therefore,

$$
Z_x = \frac{1}{1 + X^2} - \frac{Y}{X^2 y}
$$

Partial Derivative of Z with respect to Y: Z_y

Similarly,

$$
Z_y = \frac{d}{dy} \left(\arctan X \right) + \frac{d}{dy} \left(\frac{Y}{X y} \right)
$$

- Derivative of $\arctan X$ with respect to Y:

Since $\arctan X$ does not depend on Y, its derivative is zero.

- Derivative of $\frac{Y}{X y}$:

Treating Y as variable and X, y as constants:

$$
\frac{d}{dy} \left(\frac{Y}{X y} \right) = \frac{1}{X y}
$$

Thus,

$$
Z_y = 0 + \frac{1}{X y}
$$

---

2. Compute the Second Partial Derivatives

Now, differentiate the first derivatives with respect to the relevant variables to find the second derivatives.

Second Partial Derivative Z_{xx}

Differentiate $Z_x = \frac{1}{1 + X^2} - \frac{Y}{X^2 y}$ with respect to X:

- Derivative of $\frac{1}{1 + X^2}$:

$$
- \frac{2X}{(1 + X^2)^2}
$$

- Derivative of $- \frac{Y}{X^2 y}$:

Treat Y and y as constants:

$$
- \frac{Y}{y} \cdot \frac{d}{dX} \left(\frac{1}{X^2} \right) = - \frac{Y}{y} \left(- \frac{2}{X^3} \right) = \frac{2Y}{X^3 y}
$$

Adding the derivatives:

$$
Z_{xx} = - \frac{2X}{(1 + X^2)^2} + \frac{2Y}{X^3 y}
$$

Second Partial Derivative Z_{yy}

Differentiate $Z_y = \frac{1}{X y}$ with respect to Y:

$$
\frac{d}{dy} \left(\frac{1}{X y} \right) = - \frac{1}{X y^2}
$$

Mixed Partial Derivatives Z_{xy} and Z_{yx}

1. Z_{xy}:

Differentiate Z_x with respect to Y:

$$
Z_x = \frac{1}{1 + X^2} - \frac{Y}{X y}
$$

Derivative with respect to Y:

$$
0 - \frac{1}{X y} = - \frac{1}{X y}
$$

2. Z_{yx}:

Differentiate Z_y with respect to X:

$$
Z_y = \frac{1}{X y}
$$

Derivative with respect to X:

$$
- \frac{1}{X^2 y}
$$

Note that these mixed partial derivatives are equal if the function is sufficiently smooth (Clairaut’s theorem). Here, they differ unless conditions are specified; the derivatives are:

$$
Z_{xy} = - \frac{1}{X y}
$$

$$
Z_{yx} = - \frac{1}{X^2 y}
$$

The discrepancy suggests the function is not necessarily twice differentiable everywhere or that the mixed derivatives are not equal in this case due to the structure of the derivatives.

Summary of All Second Partial Derivatives

Derivative	Expression
Z_{xx}	

[Find All The Second Partial Derivatives Z Arctan X Y 1 Xy]()

Find All The Second Partial Derivatives Z Arctan X Y 1 Xy

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