Find The Area Of The Region. Inside R = 2a Cos() And Outside R = A

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Understanding how to determine the area of a region bounded by different curves is a fundamental concept in calculus and geometry. In this article, we will explore how to find the area of the region enclosed inside the polar curve R = 2a Cos(□) and outside the circle R = a. This problem combines the principles of polar coordinates, curve analysis, and integration, providing a comprehensive example of applying calculus to geometric figures.

Understanding the Curves Involved

Before we delve into the calculations, it's essential to understand the nature of the curves involved:

1. The Polar Rose: R = 2a Cos(□)

- Shape and Behavior:
The curve R = 2a Cos(□) describes a lemniscate-like shape symmetric about the polar axis, often known as a cardioid when in certain forms. Specifically, for □ in [0, 2□], the curve traces out a heart-shaped figure with a maximum radius of 2a at □ = 0 and a minimum radius of -2a at □ = □.

- Key Features:
- Maximum radius at □ = 0 is R = 2a.

- The curve passes through the origin at $\theta = \pi/2$ and $3\pi/2$.
- Symmetric about the polar axis ($\theta = 0$ line).

2. The Circle: R = a

- Shape and Behavior:

This is a circle centered at the origin with radius a. It includes all points that satisfy R = a regardless of θ.

- Position Relative to the Polar Axis:

The circle is symmetric about the origin and encompasses all points with a distance a from the center.

Determining the Region of Interest

The goal is to find the area of the region that is inside the lemniscate R=2a Cos(θ) but outside the circle R=a. To do this, we need to:

- Find the angular range(s) where the lemniscate's radius exceeds the circle's radius, i.e., where:

R_{lemniscate} = 2a Cos(θ) $\geq$ a = R_{circle}.

- Identify the intersection points between the two curves to set proper limits for integration.

Finding the Intersection Points

To determine the limits of integration, solve for $\square$ where the two curves intersect:

$$
2a \cos(\square) = a
$$

Dividing both sides by a:

$$
2 \cos(\square) = 1
$$

$$
\cos(\square) = \frac{1}{2}
$$

The solutions to this are:

$$
\square = \pm \frac{\pi}{3} + 2\square k,\quad k \in \mathbb{Z}
$$

Considering the principal values within [0, 2$\square$]:

$$
\square = \frac{\pi}{3},\quad 2\square - \frac{\pi}{3} = \frac{5\pi}{3}
$$

So, the curves intersect at:

- $\theta = \frac{\pi}{3}$
- $\theta = \frac{5\pi}{3}$

Identifying the Region to Integrate

Between these intersection points, the lemniscate lies outside the circle R = a, because:

$$
2a \cos(\theta) \geq a \Rightarrow \cos(\theta) \geq \frac{1}{2}
$$

which is true for:

$$
-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}
$$

and

$$
\frac{5\pi}{3} \leq \theta \leq 2\pi
$$

However, since the lemniscate is symmetric about the polar axis and the circle is centered at the origin, the main regions of interest are between $\theta = \pi/3$ and $\theta = 5\pi/3$, where the lemniscate exceeds the circle.

But, to find the area outside the circle but inside the lemniscate, we need to:

- Restrict the integration to the angular range where the lemniscate's radius is greater than the circle's radius, i.e., where:

$$
2a \cos(\square) \geq a \Rightarrow \cos(\square) \geq \frac{1}{2}
$$

- Recognize that the region is symmetric about the axis, simplifying calculations.

Setting Up the Area Integral

The general formula for the area enclosed by a polar curve $R(\square)$ between $\square = \square$ and $\square = \square$ is:

$$
A = \frac{1}{2} \int_{\square}^{\square} [R(\square)]^2 d\square
$$

In our case, the region of interest is where:

- The lemniscate's radius is greater than the circle's radius (i.e., outside the circle), specifically in the ranges between the intersection points.

- To find the area inside the lemniscate but outside the circle, we can compute the difference:

$$
A_{region} = \text{Area inside } R=2a \cos(\square) \text{ between } \square=\square/3 \text{ and } \square=5\square/3 \,
\text{minus} \, \text{area inside } R=a \text{ over the same range}
$$

This involves two integrals:

$$
A_{total} = \frac{1}{2} \int_{\theta_1}^{\theta_2} [(2a \cos(\theta))^2 - a^2] d\theta
$$

where $\theta_1 = \pi/3$ and $\theta_2 = 5\pi/3$.

Calculating the Area

Let's proceed step-by-step:

1. Area inside the lemniscate between the intersection points:

$$
A_{lemniscate} = \frac{1}{2} \int_{\pi/3}^{5\pi/3} [2a \cos(\theta)]^2 d\theta = \frac{1}{2} \int_{\pi/3}^{5\pi/3} 4a^2 \cos^2(\theta) d\theta
$$

$$
A_{lemniscate} = 2a^2 \int_{\pi/3}^{5\pi/3} \cos^2(\theta) d\theta
$$

Recall that:

$$
\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}
$$

\]

So,

\[
A_{lemniscate} = 2a^2 \int_{□/3}^{5□/3} \frac{1 + \cos(2□)}{2} d□ = a^2 \int_{□/3}^{5□/3} [1 + \cos(2□)] d□
\]

Integrate term-by-term:

\[
A_{lemniscate} = a^2 \left[\int_{□/3}^{5□/3} 1 d□ + \int_{□/3}^{5□/3} \cos(2□) d□ \right]
\]

Calculate each integral:

- \(\int_{□/3}^{5□/3} 1 d□ = \left[□ \right]_{□/3}^{5□/3} = \frac{5□}{3} - \frac{□}{3} = \frac{4□}{3}\)

- \(\int_{□/3}^{5□/3} \cos(2□) d□ = \left[\frac{\sin(2□)}{2} \right]_{□/3}^{5□/3}\)

Evaluate \(\sin(2□)\):

At □ = 5□/3:

\[
2□ = 10□/3 \Rightarrow \sin(10□/3) = \sin(10□/3 - 2□ \times 3) = \sin(10□/3 - 6□) = \sin(10□/3 - 6□)
\]

Note:

$$
6\square = 18\square/3
$$

So,

$$
10\square/3 - 18\square/3 = -8\square/3
$$

$\sin(-8\square/3) = -\sin(8\square/3)$

$\sin(8\square/3) = \sin(2\square + 2\square/3) = \sin(2\square) \cos(2\square/3) + \cos(2\square) \sin(2\square/3) = 0 \times \cos(2\square/3) + 1 \times \sin(2\square/3) = \sin(2\square/3) = \sqrt{3}/2$

Therefore,

$$
\sin(10\square/3) = - \sqrt{3}/2
$$

Similarly, at □ = □/3:

$$
2\square = 2\square
$$

Frequently Asked Questions

How do you find the area of the region bounded by the polar curve R = 2a cos(□) and the circle R = a?

First, identify the points of intersection of the two curves by solving 2a cos(□) = a. Then, set up the integral for the area inside the polar curve but outside the circle, typically subtracting the circle's area from the polar region's area within the intersection limits. Use the formula for polar area: (1/2)□ (R(□))^2 d□ over the relevant interval.

What is the intersection point between R = 2a cos(□) and R = a in polar coordinates?

Set 2a cos(□) = a, which simplifies to cos(□) = 1/2. Therefore, the curves intersect at □ = ±□/3.

How do the shapes of R = 2a cos(□) and R = a differ in the plane?

R = 2a cos(□) is a lemniscate-like polar curve that forms a cardioid shape, with maximum radius 2a at □ = 0, while R = a is a circle of radius a centered at the origin. The lemniscate extends beyond the circle in certain regions.

How can symmetry be used to simplify the area calculation between these two regions?

Since both curves are symmetric about the polar axis (□=0), you can compute the area in one quadrant and multiply by 2 to find the total area, simplifying the integral calculations.

What is the significance of the parameter 'a' in determining the area between these two regions?

The parameter 'a' determines the size of the circle R = a and influences the shape and size of the lemniscate R = 2a cos(□). As 'a' changes, the intersection points and the enclosed area change proportionally, allowing for scalable area calculations.

Additional Resources

Find The Area Of The Region Inside R = 2a Cos(□) And Outside R = a

Understanding the area of regions defined by polar equations is a fundamental aspect of mathematical analysis, especially in the fields of calculus and geometry. When dealing with curves described in polar coordinates, the process involves integrating the area element over the specified region. In this detailed exploration, we focus on the region bounded by the polar curve R = 2a Cos(□) and the circle R = a, aiming to compute the area that lies inside the first and outside the second.

Introduction to Polar Curves and Area Calculation

Polar coordinates are a way of representing points in the plane using a radius (distance from the origin) and an angle. The typical notation is (R, □), where:

- R is the distance from the origin to the point,
- □ (theta) is the angle measured from the positive x-axis.

Polar equations describe curves via relationships involving R and □. For example:

- R = constant describes a circle centered at the origin.
- R = a cos(□) or R = a sin(□) describes limacons or cardioids.

Calculating areas bounded by polar curves involves integrating the differential area element in polar coordinates:

\[

A = \frac{1}{2} \int_{\theta_1}^{\theta_2} R^2 \, d\theta

\]

This formula accounts for the sector of the circle between and , with R varying along the curve.

Understanding the Given Curves

The region of interest is bounded inside R = 2a Cos() and outside R = a.

Let's analyze each:

2a Cos()

- This is a lemniscate-shaped or cardioid-like curve depending on the value of a.
- For in [0,], R varies between 0 and 2a.
- The maximum R occurs at = 0, where R = 2a.
- The curve passes through the origin at = /2 (since Cos(/2) = 0).

R = a

- This is a circle centered at the origin with radius a.
- The circle is symmetric about the origin.

Identifying the Region of Interest

The task is to find the area of the region that lies inside R = 2a Cos(□) and outside R = a.

Key points:

- The region is bounded above by the curve R = 2a Cos(□).
- The region is bounded below by the circle R = a.
- The intersection points between the two curves determine the limits of integration.

Finding the Intersection Points

To find where the curves intersect, set R equal for both:

$$
2a \cos(\theta) = a
$$

Divide both sides by a (assuming a □ 0):

$$
2 \cos(\theta) = 1
$$

$$
\cos(\theta) = \frac{1}{2}
$$

Recall that:

$$
\cos(\theta) = \frac{1}{2} \quad \Rightarrow \quad \theta = \pm \frac{\pi}{3} + 2k\pi
$$

Within the principal range $[0, 2\pi]$, the solutions are:

$$
\theta = \frac{\pi}{3} \quad \text{and} \quad \theta = \frac{5\pi}{3}
$$

These are the points of intersection between the two curves, corresponding to the angles where the boundary of the region switches from being inside $R = 2a \, \text{Cos}(\theta)$ to outside.

Determining the Region's Angular Limits

Since the curves are symmetric about the polar axis:

- The curve $R = 2a \, \text{Cos}(\theta)$ reaches its maximum at $\theta = 0$.
- The intersection points at $\theta = \pi/3$ and $5\pi/3$ define the limits for integration.

The region of interest is thus between $\theta = -\pi/3$ and $\theta = \pi/3$, or equivalently, between $\theta = 0$ and $\theta = \pi/3$, considering symmetry, and then doubled.

Formulating the Area Integral

To compute the area inside R = 2a Cos(□) and outside R = a, we consider:

$$
\boxed{
A = \text{Area inside } R=2a \cos \theta \text{ and outside } R=a
}
$$

This can be expressed as:

$$
A = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left[R_{outer}^2 - R_{inner}^2 \right] d\theta
$$

Where:

- R_{outer} is the boundary curve that encloses the region from outside, which in this case is $R = 2a \cos \theta$.
- R_{inner} is the boundary curve from inside, which is $R = a$.

The limits θ_1 and θ_2 are from $-\pi/3$ to $\pi/3$.

Calculating the Area Step-by-Step

Step 1: Set up the integral

$$
A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left[(2a \cos \theta)^2 - a^2 \right] d\theta
$$

Step 2: Simplify the integrand

$$
(2a \cos \theta)^2 = 4a^2 \cos^2 \theta
$$

So the integral becomes:

$$
A = \frac{1}{2} \int_{-\pi/3}^{\pi/3} \left[4a^2 \cos^2 \theta - a^2 \right] d\theta
$$

Factor out a^2:

$$
A = \frac{a^2}{2} \int_{-\pi/3}^{\pi/3} \left[4 \cos^2 \theta - 1 \right] d\theta
$$

Step 3: Use symmetry

Since the integrand is an even function ($\cos^2(\theta)$ is even), and the limits are symmetric about zero:

$$
A = a^2 \int_{0}^{\pi/3} \left[4 \cos^2 \theta - 1 \right] d\theta
$$

Evaluating the Integral

Step 4: Simplify the integral further

$$
A = a^2 \left[\int_0^{\pi/3} 4 \cos^2 \theta \, d\theta - \int_0^{\pi/3} 1 \, d\theta \right]
$$

Calculate each integral separately.

Step 5: Use the identity for $\cos^2 \theta$

Recall:

$$
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}
$$

Thus:

$$
4 \cos^2 \theta = 4 \times \frac{1 + \cos 2\theta}{2} = 2 (1 + \cos 2\theta)
$$

Now the area becomes:

$$
A = a^2 \left[\int_0^{\pi/3} 2 (1 + \cos 2\theta) \, d\theta - \int_0^{\pi/3} 1 \, d\theta \right]
$$

Completing the Integration

Step 6: Integrate term-by-term

$$
A = a^2 \left[2 \int_0^{\pi/3} 1 \, d\theta + 2 \int_0^{\pi/3} \cos 2\theta \, d\theta - \int_0^{\pi/3} 1 \, d\theta \right]
$$

Simplify:

$$
A = a^2 \left[(2 - 1) \int_0^{\pi/3} 1 \, d\theta + 2 \int_0^{\pi/3} \cos 2\theta \, d\theta \right]
$$

$$
A = a^2 \left[\int_0^{\pi/3} 1 \, d\theta + 2 \int_0^{\pi/3} \cos 2\theta \, d\theta \right]
$$

Step 7: Perform the integrals

- First integral:

$$
\int_0^{\pi/3} 1 \, d\theta = \left[\theta \right]_0^{\pi/3} = \frac{\pi}{3}
$$

- Second integral:

\[
\int_0^{\pi/3} \cos 2\theta \, d\theta = \left[\frac{\sin

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