Find The Area Of The Region Inside: R = 9sin But Outside: R =1

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Understanding how to calculate the area of regions defined by polar equations is a fundamental concept in calculus and analytical geometry. The problem at hand involves determining the area of the region that lies inside the polar curve $R = 9\sin\theta$ but outside the circle $R = 1$. This task requires a thorough analysis of both curves, their intersections, and the appropriate integration methods to accurately compute the area. In this comprehensive guide, we will explore the step-by-step process to find this area, including the mathematical background, detailed calculations, and visual interpretations to aid understanding.

Understanding the Given Curves

To accurately determine the area of the specified region, it's essential to understand the nature and behavior of both curves involved.

The Polar Equation $R = 9\sin\theta$

- Type of Curve: This is a lemniscate or cardioid-like curve, depending on the context. In this case, $R = 9\sin\theta$ describes a limacon with a maximum radius of 9 when $\sin\theta = 1$.
- Shape and Symmetry: The curve is symmetric about the vertical axis ($\theta = \pi/2$) because sine is symmetric about this axis.
- Key Points:
- When $\theta = 0$, $R = 0$.
- When $\theta = \pi/2$, $R = 9$.
- When $\theta = \pi$, $R = 0$.
- When $\theta = 3\pi/2$, $R = -9$, which in polar coordinates corresponds to a point in the opposite direction.

The Circle $R = 1$

- Type of Curve: A simple circle with radius 1 centered at the origin.
- Position: Since R is constant at 1, it forms a circle of radius 1 around the origin, covering all angles θ from 0 to 2π.

Visualizing the Curves

Before delving into calculations, visualizing these curves helps understand the region of interest.

- The circle $R = 1$ is straightforward, a circle centered at the origin with radius 1.
- The curve $R = 9\sin\theta$:
- Touches the origin at $\theta = 0$ and $\theta = \pi$.
- Reaches its maximum radius of 9 at $\theta = \pi/2$.
- Is symmetric about the vertical axis.
- The intersection points between these two curves define the limits of the region where the area calculation is focused.

Finding Intersection Points

The first step in calculating the area involves identifying where the two curves intersect because the region of interest is bounded by these points.

Setting $R = 9\sin\theta$ equal to $R = 1$

To find the intersection points:

1. Set the two equations equal:

$9\sin\theta = 1$

2. Solve for θ:

$\sin\theta = 1/9$

3. Determine θ values:

$\theta = \arcsin(1/9)$

Since $\sin\theta = 1/9$ is positive and less than 1, there are two solutions in $[0, 2\pi]$:

- $\theta_1 = \arcsin(1/9)$
- $\theta_2 = \pi - \arcsin(1/9)$

Let’s denote:

- $\alpha = \arcsin(1/9)$

Because $1/9 \approx 0.111...$, α is approximately 0.111 radians (~6.36 degrees).

Thus, the intersection points occur at:

- $\theta = \alpha$
- $\theta = \pi - \alpha$

Determining the Region for Area Calculation

The region inside $R = 9\sin\theta$ but outside $R = 1$ is bounded between the angles $\theta = \alpha$ and $\theta = \pi - \alpha$.

- For θ in $[\alpha, \pi - \alpha]$, $R = 9\sin\theta$ is outside the circle $R = 1$.
- For θ outside these values, $R = 1$ is outside or not relevant to the region of interest.

Visual confirmation reveals that:

- Between $\theta = \alpha$ and $\theta = \pi - \alpha$, $R = 9\sin\theta > 1$.
- Outside this interval, either $R = 1$ is larger or the curves do not enclose the region.

Hence, the area calculation involves integrating over θ from α to $\pi - \alpha$.

Calculating the Area of the Region

The area enclosed between two polar curves $R_1(\theta)$ and $R_2(\theta)$, where $R_1(\theta) \geq R_2(\theta)$ over the interval, is given by:

\[\text{Area} = \frac{1}{2} \int_{\theta_1}^{\theta_2} \left[R_1^2(\theta) - R_2^2(\theta) \right] dθ \]

In our case:

- \(R_1(\theta) = 9 \sin θ \) (the outer boundary)
- \(R_2(\theta) = 1 \) (the inner boundary)

The limits are from $\theta = \alpha$ to $\theta = \pi - \alpha$.

Thus, the area \(A\) is:

\[A = \frac{1}{2} \int_{\alpha}^{\pi - \alpha} \left[(9 \sin θ)^2 - 1^2 \right] dθ \]

Simplify the integrand:

$$ A = \frac{1}{2} \int_{\alpha}^{\pi - \alpha} \left[81 \sin^2 θ - 1 \right] dθ $$

Performing the Integration

The integral can be separated into two parts:

$$ A = \frac{1}{2} \left(81 \int_{\alpha}^{\pi - \alpha} \sin^2 θ dθ - \int_{\alpha}^{\pi - \alpha} 1 dθ \right) $$

Calculate both integrals separately.

Integral of $\sin^2 θ$

Recall the identity:

$$ \sin^2 θ = \frac{1 - \cos 2θ}{2} $$

Thus,

$$ \int \sin^2 θ dθ = \int \frac{1 - \cos 2θ}{2} dθ = \frac{1}{2} \int 1 dθ - \frac{1}{2} \int \cos 2θ dθ $$

Compute:

$$ \int \sin^2 θ dθ = \frac{θ}{2} - \frac{1}{4} \sin 2θ + C $$

Now, evaluate from θ = α to θ = π - α:

$$ \int_{α}^{\pi - α} \sin^2 θ dθ = \left[\frac{θ}{2} - \frac{1}{4} \sin 2θ \right]_{α}^{\pi - α} $$

Integral of 1 over the interval

This is straightforward:

$$ \int_{α}^{\pi - α} 1 dθ = (\pi - α) - α = \pi - 2α $$

Putting It All Together

The total area is:

$$ A = \frac{1}{2} \left[81 \left(\frac{\theta}{2} - \frac{1}{4} \sin 2\theta \right)_{\alpha}^{\pi - \alpha} - (\pi - 2\alpha) \right] $$

Simplify step-by-step:

1. Evaluate the first term:

$$ 81 \left(\frac{\pi - \alpha}{2} - \frac{1}{4} \sin 2(\pi - \alpha) \right) - 81 \left(\frac{\alpha}{2} - \frac{1}{4} \sin 2\alpha \right) $$

2. Note that:

$$ \sin 2(\pi - \alpha) = \sin (2\pi - 2\alpha) = - \sin 2\alpha $$

because $\sin (A - B) = \sin A \cos B - \cos A \sin B$, but more directly, $\sin (2\pi - x) = - \sin x$.

3. Substituting:

$$ \sin 2(\pi - \alpha) = - \sin 2\alpha $$

4. Now, the difference becomes:

$$ \frac{\pi - \alpha}{2} - \frac{1}{4} (- \sin 2\alpha) - \left(\frac{\alpha}{2} - \frac{1}{4} \sin 2\alpha \right) $$

which simplifies to:

$$ \frac{\pi - \alpha}{2} + \frac{1}{4} \sin 2\alpha - \frac{\alpha}{2} + \frac{1}{4} \sin 2\alpha $$

Combine like terms:

$$ \left(\frac{\pi - \alpha}{2} - \frac{\alpha}{2} \right) + \left(\frac{1}{4} \sin 2\alpha + \frac{1}{4} \sin 2\alpha \right) $$

$$ = \frac{\pi - 2\alpha}{2} + \frac{1}{2} \sin 2 $$

Frequently Asked Questions

What is the main goal when finding the area of the

region inside R = 9sin θ but outside R = 1?

The goal is to compute the area enclosed between the two polar curves, specifically the area inside $R = 9\sin\theta$ that is not overlapped by $R = 1$.

How do you determine the points of intersection between R = 9sin θ and R = 1?

Set $9\sin\theta = 1$ and solve for θ, which gives $\sin\theta = 1/9$; thus, $\theta = \arcsin(1/9)$.

What are the limits of integration when calculating the area between these two curves?

The limits are from $\theta = 0$ to $\theta = \arcsin(1/9)$, covering the region where $R = 9\sin\theta$ is outside $R = 1$.

What is the formula for the area enclosed between two polar curves?

The area is given by $(1/2) \int [R_{outer}(\theta)]^2 - [R_{inner}(\theta)]^2 \, d\theta$ over the relevant interval.

Which curve acts as the outer boundary in the region of interest?

$R = 9\sin\theta$ acts as the outer boundary where it exceeds $R = 1$, which is inside the region.

How do symmetry considerations simplify the calculation of the area?

Since the curves are symmetric about the line $\theta = \pi/2$, you can calculate the area over a quarter or half of the region and multiply accordingly.

What is the step-by-step method to compute the area inside R = 9sin θ but outside R = 1?

First, find the intersection points; then set up the integral of $(1/2)[(9\sin\theta)^2 - 1^2] \, d\theta$ from 0 to $\theta = \arcsin(1/9)$; finally, evaluate the integral to find the area.

Why is understanding the graph of the curves important in solving this problem?

Graphing helps visualize the region, identify intersection points, and determine which curve forms the outer boundary over each interval, guiding the setup of the integral.

Additional Resources

Find The Area Of The Region Inside: $R = 9\sin\theta$ But Outside: $R = 1$

Introduction

Mathematics, especially the realm of polar coordinates, offers a fascinating landscape for exploring geometric regions defined by various equations. Among these, rose and circle equations often produce intriguing shapes with properties that challenge our spatial intuition. This article delves into the problem of finding the area of the region inside the polar curve $R = 9 \sin \theta$ but outside the circle $R = 1$.

This problem is more than an abstract exercise; it exemplifies the practical application of polar integration, symmetry considerations, and geometric reasoning. Understanding the area enclosed by such complex regions is fundamental in fields ranging from physics (e.g., electromagnetic fields) to engineering and computer graphics.

The Geometric Context and Relevance

Polar Curves and Their Significance

Polar coordinates (R, θ) describe points in a plane via a distance R from the origin and an angle θ from a fixed reference axis. Curves expressed in polar form often produce shapes like circles, lemniscates, roses, and cardioids, each with unique properties.

The Curves in Focus

- Rose Curve: $R = 9 \sin \theta$
This is a classic rose curve with petals aligned along the vertical axis, symmetric about the origin. Its maximum radius is 9, attained at $\theta = \pi/2$.

- Circle: $R = 1$
A simple circle centered at the origin with radius 1.

The problem involves identifying the area of the region that lies inside the rose curve but outside the circle. Such a region, due to symmetry, can be efficiently analyzed by exploiting the inherent properties of these curves.

Analytical Approach to the Problem

Step 1: Understanding the Curves

The Rose Curve $R = 9 \sin \theta$

- Symmetric with respect to the $\theta = \pi/2$ axis.
- Petals extend from the origin outward to a maximum radius of 9 at $\theta = \pi/2$.
- The curve completes a full "petal" pattern over $0 \leq \theta \leq \pi$ because R is non-negative in this range for $\sin \theta \geq 0$.

The Circle $R = 1$

- Centered at the origin with radius 1.
- Entirely contained within the rose curve's inner regions for certain θ values.

Step 2: Determining the Intersection Points

We need to find where the two curves intersect, i.e., where:

$$
9 \sin \theta = 1
$$

which leads to:

$$
\sin \theta = \frac{1}{9}
$$

Since $\sin \theta$ is positive in the first and second quadrants, the solutions are:

$$
\theta = \arcsin \left(\frac{1}{9} \right) \quad \text{and} \quad \pi - \arcsin \left(\frac{1}{9} \right)
$$

Let:

$$
\theta_0 = \arcsin \left(\frac{1}{9} \right)
$$

Numerically,

$$
\theta_0 \approx \arcsin(0.1111) \approx 0.1114 \, \text{radians}

\]

The intersection points are therefore at:

$$
\theta = \theta_0 \quad \text{and} \quad \theta = \pi - \theta_0
$$

Defining the Region of Interest

Visualizing the Region

Between θ_0 and $\pi - \theta_0$:

- The rose curve satisfies $R \geq 1$, since $9 \sin \theta \geq 1$.
- Outside the circle $R=1$, the rose curve extends beyond the circle.

Thus, the region inside the rose curve but outside the circle occurs primarily in the interval:

$$
\theta \in [\theta_0, \pi - \theta_0]
$$

and symmetrically on the other side due to the symmetry of the curves.

Symmetry Considerations

- The rose curve $R=9 \sin \theta$ is symmetric about $\theta = \pi/2$.
- The region of interest is symmetric about the y-axis.
- To simplify calculations, focus on $\theta \in [\theta_0, \pi/2]$, then double the result.

Calculating the Area

General Formula in Polar Coordinates

The area A enclosed between two curves $R_1(\theta)$ and $R_2(\theta)$ over $\theta \in [a, b]$ is:

$$
A = \frac{1}{2} \int_{a}^{b} \left(R_1^2(\theta) - R_2^2(\theta) \right) d\theta
$$

In our case:

- \(R_{\text{outer}}(\theta) = 9 \sin \theta \)
- \(R_{\text{inner}}(\theta) = 1 \)

The region's area is obtained by integrating over the relevant \(\theta\) interval.

Step 1: Set up the integral

Because the region is symmetric, the total area \(A\) is:

\[
A = 2 \times \left[\frac{1}{2} \int_{\theta_0}^{\pi/2} \left((9 \sin \theta)^2 - 1^2 \right) d\theta \right]
\]

which simplifies to:

\[
A = \int_{\theta_0}^{\pi/2} \left(81 \sin^2 \theta - 1 \right) d\theta
\]

Step 2: Compute the integral

Break the integral into two parts:

\[
A = 81 \int_{\theta_0}^{\pi/2} \sin^2 \theta \, d\theta - \int_{\theta_0}^{\pi/2} 1 \, d\theta
\]

The integrals are standard:

- \(\int \sin^2 \theta \, d\theta = \frac{\theta}{2} - \frac{\sin 2 \theta}{4} + C\)

Apply limits:

\[
A = 81 \left[\frac{\theta}{2} - \frac{\sin 2 \theta}{4} \right]_{\theta_0}^{\pi/2} - \left[\theta \right]_{\theta_0}^{\pi/2}
\]

which becomes:

\[
A = 81 \left(\frac{\pi/2}{2} - \frac{\sin (\pi)}{4} - \left(\frac{\theta_0}{2} - \frac{\sin 2 \theta_0}{4} \right) \right) - \left(\frac{\pi}{2} - \theta_0 \right)
\]

Recall that \(\sin \pi = 0\), so:

\[
A = 81 \left(\frac{\pi}{4} - 0 - \frac{\theta_0}{2} + \frac{\sin 2 \theta_0}{4} \right) - \left(\frac{\pi}{2} - \theta_0 \right)
\]

Simplify:

\[
A = 81 \left(\frac{\pi}{4} - \frac{\theta_0}{2} + \frac{\sin 2 \theta_0}{4} \right) - \frac{\pi}{2} + \theta_0
\]

Step 3: Express in terms of known quantities

Recall:

\[
\theta_0 = \arcsin \left(\frac{1}{9} \right)
\]

and:

\[
\sin 2 \theta_0 = 2 \sin \theta_0 \cos \theta_0
\]

We already have:

\[
\sin \theta_0 = \frac{1}{9}
\]

To find \(\cos \theta_0\):

\[
\cos \theta_0 = \sqrt{1 - \sin^2 \theta_0} = \sqrt{1 - \frac{1}{81}} = \sqrt{\frac{80}{81}} = \frac{\sqrt{80}}{9} = \frac{4 \sqrt{5}}{9}
\]

Therefore:

\[
\sin 2 \theta_0 = 2 \times \frac{1}{9} \times \frac{4 \sqrt{5}}{9} = \frac{8 \sqrt{5}}{81}
\]

Now, plug these into the expression:

\[
A = 81 \left(\frac{\pi}{4} - \frac{\arcsin(1/9)}{2} + \frac{8 \sqrt{5}}{81 \times 4} \right) - \frac{\pi}{2

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