# Find The Jacobian Of The Transformation X = 4 U V , Y = U 2 6 V

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Introduction

In multivariable calculus, transformations between coordinate systems are essential tools for simplifying complex integrals, analyzing geometric properties, and solving differential equations. One of the fundamental concepts in understanding these transformations is the Jacobian determinant, which measures how a transformation scales areas or volumes locally. This article provides a comprehensive guide to finding the Jacobian of the transformation:

$$
X = 4 U V, \quad Y = U^2 6 V
$$

We will explore the concepts step-by-step, including the definition of the Jacobian, how to compute it for this specific transformation, and its applications in calculus and geometry.

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Understanding the Transformation

Before diving into the Jacobian calculation, it is important to understand the nature of the transformation itself.

The Transformation Equations

The transformation maps variables $(U, V)$ to $(X, Y)$, given by:

$$
X = 4 U V
$$
$$
Y = U^2 6 V
$$

Note that the second equation appears to be expressed as $Y = U^2 6 V$. To clarify, it's common to interpret this as:

$$
Y = 6 U^2 V
$$

assuming the notation signifies multiplication. If the original expression was meant differently, please clarify. For the purpose of this article, we will proceed with:

$$
Y = 6 U^2 V
$$

Thus, the transformation is:

$$
\boxed{
\begin{cases}
X = 4 U V \\
Y = 6 U^2 V
\end{cases}
}
$$

This transformation involves products and powers of $U$ and $V$, making the Jacobian an important tool to analyze how areas scale under the transformation.

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## The Concept of the Jacobian

### Definition

The Jacobian determinant of a transformation from $(U, V)$ to $(X, Y)$ is given by:

$$
J = \det \left( \frac{\partial (X, Y)}{\partial (U, V)} \right)
$$

which is computed as:

$$
J =
\begin{vmatrix}
\frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\
\frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V}
\end{vmatrix}
$$

The Jacobian determinant indicates how an infinitesimal area element $dU \, dV$ in the $(U, V)$ plane maps to an area element $dX \, dY$ in the $(X, Y)$ plane:

$$
dX \, dY = |J| \, dU \, dV
$$

This is critical when changing variables in multiple integrals.

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## Step-by-Step Calculation of the Jacobian

### Step 1: Compute the Partial Derivatives

For the transformation:

$$
X = 4 U V
$$

$$
Y = 6 U^2 V
$$

calculate the partial derivatives:

$$
\frac{\partial X}{\partial U}, \quad \frac{\partial X}{\partial V}
$$
$$
\frac{\partial Y}{\partial U}, \quad \frac{\partial Y}{\partial V}
$$

Let's compute each:

- $\frac{\partial X}{\partial U}$:

$$
\frac{\partial}{\partial U} (4 U V) = 4 V
$$

- $\frac{\partial X}{\partial V}$:

$$
\frac{\partial}{\partial V} (4 U V) = 4 U
$$

- $\frac{\partial Y}{\partial U}$:

$$
\frac{\partial}{\partial U} (6 U^2 V) = 12 U V
$$

- $\frac{\partial Y}{\partial V}$:

$$
\frac{\partial}{\partial V} (6 U^2 V) = 6 U^2
$$

Step 2: Write the Jacobian Matrix

Construct the matrix with these derivatives:

$$
J_{matrix} =
\begin{bmatrix}
\frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\
\frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V}
\end{bmatrix}
=
\begin{bmatrix}
4 V & 4 U \\
12 U V & 6 U^2
\end{bmatrix}
$$

Step 3: Calculate the Determinant

The Jacobian determinant $J$ is:

$$
J = (4 V)(6 U^2) - (4 U)(12 U V)
$$

Simplify each term:

- $4 V \times 6 U^2 = 24 U^2 V$
- $4 U \times 12 U V = 48 U^2 V$

Thus,

$$
J = 24 U^2 V - 48 U^2 V = -24 U^2 V
$$

---

Final Expression for the Jacobian

$$
\boxed{
\text{Jacobian } J(U, V) = -24 U^2 V
}
$$

The absolute value of the Jacobian determinant indicates how areas are scaled:

$$
|J| = 24 U^2 |V|
$$

This scalar function depends on $U$ and $V$, which implies local area scaling varies across the domain.

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Geometric Interpretation

The Jacobian determinant describes how an infinitesimal area in the $(U, V)$ plane maps to an area in the $(X, Y)$ plane:

- When $|J| > 1$, the transformation locally enlarges areas.
- When $|J| < 1$, the transformation reduces the area.
- When $|J| = 0$, the transformation is singular at that point, indicating a possible collapse of areas or a critical point.

In this specific case, the Jacobian is zero when:

$$
U = 0 \quad \text{or} \quad V = 0
$$

meaning the transformation is singular along the lines $U=0$ and $V=0$.

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Applications of the Jacobian in Integral Calculations

The Jacobian is essential in changing variables in multiple integrals. For example, suppose we want to evaluate an integral over a region in the $(X, Y)$ plane. By transforming to $(U, V)$, the integral becomes:

$$
\iint_{R_{XY}} f(X, Y) \, dX \, dY = \iint_{R_{UV}} f(4 U V, 6 U^2 V) |J(U, V)| \, dU \, dV
$$

where $R_{UV}$ is the pre-image of the region $R_{XY}$.

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Example: Computing an Integral Using the Jacobian

Suppose we want to compute the integral:

$$
I = \iint_{D} X \, dX \, dY
$$

over a region $D$ in the $(X, Y)$ plane that maps from a simple region in $(U, V)$ space. Using the transformation and Jacobian:

$$
I = \iint_{D_{UV}} (4 U V) \times |J(U, V)| \, dU \, dV
$$

which simplifies to:

$$
I = \iint_{D_{UV}} 4 U V \times 24 U^2 |V| \, dU \, dV = 96 U^3 V |V| \, dU \, dV
$$

The explicit limits depend on the region $D_{UV}$, which should be chosen based on the problem context.

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Summary and Key Takeaways

- The transformation $X = 4 U V, \quad Y = 6 U^2 V$ maps the $(U, V)$ plane to the $(X, Y)$ plane.
- The Jacobian determinant measures how areas scale under this transformation.
- The Jacobian for this transformation is:

$$
J(U, V) = -24 U^2 V
$$

- Sign and magnitude of $J$ provide insight into local area distortion and singular points.
- The Jacobian plays a crucial role in changing variables in multiple integrals, facilitating easier computation of complex regions.

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Final Notes

Understanding how to compute the Jacobian determinant for various transformations enables mathematicians and engineers to analyze complex systems, evaluate integrals over transformed regions, and interpret geometric properties. The method demonstrated here can be extended to higher dimensions, more complex transformations, and different types of coordinate changes such as polar, cylindrical, or spherical coordinates.

If you encounter transformations involving products, powers, or other nonlinear operations, computing the Jacobian follows similar steps: find partial derivatives, form the matrix, and compute its determinant.

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Additional Resources

- Multivariable Calculus Textbooks: For deeper understanding of Jacobians and coordinate transformations.
- Online Calculus Tutorials: Visualizations of how transformations affect areas and volumes.
- Mathematical Software

# Frequently Asked Questions

## How do you find the Jacobian of the transformation X = 4 U V and Y = U^2 - 6 V?

To find the Jacobian, compute the determinant of the matrix of partial derivatives: $J = |\partial(X,Y)/\partial(U,V)|$. That is, $J = |(\partial X/\partial U, \partial X/\partial V); (\partial Y/\partial U, \partial Y/\partial V)|$. Calculate each derivative and then find the determinant.

## What are the partial derivatives needed to compute the Jacobian for the transformation X=4UV and Y=U^2 - 6V?

The partial derivatives are: $\partial X/\partial U = 4V$, $\partial X/\partial V = 4U$, $\partial Y/\partial U = 2U$, and $\partial Y/\partial V = -6$. These form the Jacobian matrix.

## What is the formula for the Jacobian determinant in the given transformation?

The Jacobian determinant is $J = (\partial X/\partial U)(\partial Y/\partial V) - (\partial X/\partial V)(\partial Y/\partial U)$. Substituting the derivatives gives $J = (4V)(-6) - (4U)(2U) = -24V - 8U^2$.

## How does the Jacobian determinant relate to changing variables in multiple integrals for the transformation X=4UV and Y=U^2 - 6V?

The Jacobian determinant provides the scaling factor when changing variables in integrals. Specifically, the differential area element $dX\, dY = |J|\, dU\, dV$, where $|J|$ is the absolute value of the Jacobian determinant.

### Are there any special considerations or restrictions when computing the Jacobian for this transformation?

Yes, the Jacobian determinant should not be zero, as this indicates the transformation is locally invertible. Points where $J=0$ may lead to singularities or non-invertible regions, which need to be examined depending on the context.

## Additional Resources

Jacobian of Transformation: A Deep Dive into Calculus and Coordinate Transformations

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When exploring the fascinating world of multivariable calculus, one of the most fundamental and powerful tools you encounter is the Jacobian determinant. Whether you're transforming variables for integration, solving complex systems, or understanding how functions change under various coordinate mappings, mastering the Jacobian is essential. Today, we'll dissect the process of finding the Jacobian for a specific transformation:

$$
X = 4 U V \quad \text{and} \quad Y = U^2 - 6 V
$$

This example offers an excellent opportunity to understand the mechanics behind Jacobian calculations, their significance, and practical applications. Think of it as a detailed product review—but instead of gadgets, we're reviewing the transformation process itself.

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## Understanding the Transformation: Context and Significance

Before diving into calculations, it’s crucial to grasp what this transformation represents and why the Jacobian matters.

The Transformation Defined

The functions:

$$
X = 4 U V
$$
$$
Y = U^2 - 6 V
$$

map points from the $(U, V)$ plane to the $(X, Y)$ plane. Think of this as a "coordinate change," often used to simplify integrals or analyze functions more conveniently in a different coordinate system.

Why Is the Jacobian Important?

The Jacobian determinant, often denoted as $\frac{\partial (X, Y)}{\partial (U, V)}$, measures how infinitesimal areas (or volumes in higher dimensions) are scaled under the transformation. When performing double integrals, for example, the Jacobian adjusts the $dU\,dV$ area element to match the $dX\,dY$ element:

$$
dX\,dY = |\text{Jacobian}| \times dU\,dV
$$

A Jacobian close to zero indicates the transformation compresses areas, while a large magnitude signals expansion. If the Jacobian is negative, it also indicates a change in orientation.

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# Step-by-Step Calculation of the Jacobian

The process of finding the Jacobian involves calculating the determinant of the matrix of partial derivatives:

$$
J = \frac{\partial (X,Y)}{\partial (U,V)} =
\begin{bmatrix}
\frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\
\frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V}
\end{bmatrix}
$$

The determinant of this matrix provides the Jacobian value.

Step 1: Compute the Partial Derivatives

Let's compute each component carefully:

- $\frac{\partial X}{\partial U}$

$$
X = 4 U V
$$
$$
\frac{\partial X}{\partial U} = 4 V
$$

- $\frac{\partial X}{\partial V}$

$$
\frac{\partial X}{\partial V} = 4 U
$$

- $\frac{\partial Y}{\partial U}$

$$
Y = U^2 - 6 V
$$

$$
\frac{\partial Y}{\partial U} = 2 U
$$

- $\frac{\partial Y}{\partial V}$

$$
\frac{\partial Y}{\partial V} = -6
$$

Step 2: Assemble the Jacobian Matrix

Putting the derivatives into the matrix:

$$
J = \begin{bmatrix}
4 V & 4 U \\
2 U & -6
\end{bmatrix}
$$

Step 3: Calculate the Determinant

The determinant of a 2x2 matrix:

$$
\det \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = ad - bc
$$

Applying this:

$$
J = (4 V)(-6) - (4 U)(2 U) = -24 V - 8 U^2
$$

Final Jacobian:

$$
\boxed{
J(U, V) = -24 V - 8 U^2
}
$$

This expression encapsulates how the area element transforms under the given coordinate change.

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# Interpreting the Jacobian Expression

The derived Jacobian:

$$
J(U, V) = -24 V - 8 U^2

\]

has several notable features:

1. Dependence on $U$ and $V$:
The Jacobian varies across the domain, implying the transformation scales areas differently at different points.

2. Negative Significance:
The negative indicates a change in orientation—areas are "flipped" under the transformation at certain points.

3. Zeroes of the Jacobian:
Setting $J(U, V) = 0$:

$$
-24 V - 8 U^2 = 0 \implies 3 V + U^2 = 0
$$

These points form a curve in the $(U, V)$ plane where the transformation locally "degenerates," meaning the area element shrinks to zero, often corresponding to singularities or boundaries in the transformed domain.

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# Applications and Practical Considerations

Understanding the Jacobian for this transformation is crucial in various contexts:

1. Changing Variables in Double Integrals

Suppose you're asked to evaluate an integral over a region in the $(X, Y)$ plane. Transforming the integral into $(U, V)$ coordinates simplifies the region or the integrand itself. The integral transforms as:

$$
\iint_{R_{XY}} f(X, Y) \, dX\, dY = \iint_{R_{UV}} f(4 U V, U^2 - 6 V) \times |J(U, V)| \, dU\, dV
$$

Knowing $J(U, V)$ allows precise calculation of the area scaling.

2. Analyzing Critical Points and Singularities

The points where $J(U, V) = 0$ are critical because they signal where the transformation loses invertibility or where the mapped region might fold or collapse—crucial in advanced calculus and differential geometry.

3. Mapping Domains and Visualizing Transformations

Understanding how the domain in $(U, V)$ maps to $(X, Y)$ helps visualize complex transformations, especially when modeling physical phenomena like fluid flows or electromagnetic fields.

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## Special Cases and Further Insights

- When $V = 0$:
The Jacobian simplifies to $-8 U^2$, which is always non-positive, becoming zero when $U=0$.
- Implication: Along the line $V=0$, the transformation's local area scale depends solely on $U$. When $U=0$, there's a degeneracy.

- When $U=0$:
The Jacobian reduces to $-24 V$.
- Implication: Along $U=0$, the area's scaling depends linearly on $V$.

- Sign of the Jacobian:

- For $V > -\frac{U^2}{3}$, the Jacobian is negative.
- For $V < -\frac{U^2}{3}$, it becomes positive.

The change in sign indicates orientation reversal across the curve $3 V + U^2 = 0$.

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## Conclusion: Mastering the Jacobian for Complex Transformations

Calculating the Jacobian for the transformation $X=4 U V$ and $Y=U^2 - 6 V$ exemplifies the elegance and utility of multivariable calculus tools. This process not only reveals how infinitesimal areas are scaled and oriented but also lays the groundwork for advanced applications in physics, engineering, and mathematics.

By methodically deriving the partial derivatives, assembling the Jacobian matrix, and analyzing its determinant, you gain deep insight into the nature of the transformation—its strengths, limitations, and critical points. Whether you're performing a change of variables to evaluate integrals or exploring the geometry of mappings, understanding the Jacobian remains an indispensable skill.

In summary, this detailed exploration underscores that the Jacobian isn't just a mathematical formality; it's a window into the transformation's soul, revealing how space itself morphs under coordinate changes—and opening doors to solving real-world problems with confidence and precision.

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