If The Power Series An(x-4)^n Converges At X=7 And Diverges At X=9

If The Power Series An(x-4)^n Converges At X=7 And Diverges At X=9, this statement provides crucial information about the behavior of the series and its radius of convergence. Understanding what this implies requires a deep dive into the concepts of power series, convergence criteria, and how to determine the interval of convergence. In this article, we will explore these ideas systematically, elucidating the significance of convergence at specific points, the calculation of the radius of convergence, and the behavior of the series at the boundary points.

Understanding Power Series and Their Convergence

What Is a Power Series?

A power series is an infinite sum of the form:

\[\sum_{n=0}^\infty a_n (x - c)^n \]

where:
- \(a_n \) are coefficients,
- \(c \) is the center of the series,
- \(x \) is the variable.

This series represents a function that can be expanded around the point \(c \), often used to approximate functions locally.

Radius and Interval of Convergence

The key concepts associated with power series are:
- Radius of convergence (R): The distance from the center \(c \) within which the series converges.
- Interval of convergence: The actual set of \(x \)-values for which the series converges, which includes the endpoints if the series converges there.

The radius \(R \) is typically found using the Ratio Test or Root Test and is crucial in understanding the behavior of the series.

Analyzing the Given Series: Convergence at \(

x=7 \) and Divergence at \(x=9 \)

Identifying the Center and Radius of Convergence

In the series:

\[\sum_{n=0}^\infty a_n (x - 4)^n \]

the center is at \(c = 4 \). The problem states:
- The series converges at \(x=7 \),
- The series diverges at \(x=9 \).

Since the series is expressed as \((x - 4)^n \), the distance from the center to these points is:

- \(|7 - 4| = 3 \)
- \(|9 - 4| = 5 \)

The fact that the series converges at \(x=7 \) (distance 3) and diverges at \(x=9 \) (distance 5) suggests that the radius of convergence \(R \) is at least 3 but less than 5, since convergence occurs at the former point but not at the latter.

Determining the Radius of Convergence

Given these conditions, the radius of convergence \(R \) can be inferred as:

\[R = 3 \]

because the series converges at the point \(x=7 \), which is exactly \(3 \) units from the center, and diverges at \(x=9 \), which is \(5 \) units away, exceeding the radius.

This reasoning aligns with the properties of power series, where convergence at a boundary point depends on the specific series, but divergence outside the radius indicates that the radius cannot be larger than the convergence boundary at \(x=7 \).

Implications of the Convergence Behavior

Interval of Convergence

Based on the above analysis, the interval of convergence is:

\[(4 - R, 4 + R) = (1, 7) \]

since:
- The series converges at \(x=7 \),

- It diverges at $x=9$, which lies outside the radius.

However, whether the series converges at the endpoints $x=1$ and $x=7$ depends on the behavior of the series at those points. Here, it converges at $x=7$, so the interval includes the endpoint at 7, but we need to verify if it converges at $x=1$.

Behavior at the Endpoints

- At $x=1$: The distance from the center is $|1 - 4|=3$, equal to the radius, so this is a boundary point. To determine whether the series converges at $x=1$, one must analyze the series at that point, often by applying the Alternating Series Test or other convergence tests.

- At $x=7$: The series converges, as given, so the endpoint is included in the interval of convergence.

In conclusion, the interval of convergence is:

$$[1, 7]$$

assuming the series converges at $x=1$, which is typical in many practical cases, but needs confirmation.

Practical Applications and Summary

Applications of Power Series Convergence Analysis

Understanding the convergence properties of power series is vital in numerous fields:
- Mathematics: Deriving Taylor and Maclaurin series for functions.
- Physics: Modeling physical phenomena with series expansions.
- Engineering: Signal processing and control systems analysis.

Summary of Key Points

- The convergence at $x=7$ and divergence at $x=9$ indicates a radius of convergence $R=3$.
- The interval of convergence is centered at $c=4$, stretching from 1 to 7.
- Endpoint behavior must be checked individually to confirm whether they are included.
- Power series convergence behavior helps determine the domain where the series accurately represents a function.

Conclusion

The given conditions of convergence at $x=7$ and divergence at $x=9$ precisely define the radius of convergence as 3 units from the center $c=4$. This understanding is fundamental in utilizing power series effectively, whether for approximation, solving differential equations, or analyzing functions. Recognizing the importance of boundary points and their convergence behavior allows mathematicians and scientists to define the most accurate domain of a power series representation, ensuring the correct application of these mathematical tools across various disciplines.

Frequently Asked Questions

What does it mean for a power series to converge at x=7 and diverge at x=9?

It means that when substituting x=7 into the power series, the sum approaches a finite value, indicating convergence. Conversely, at x=9, the sum does not approach a finite value, indicating divergence.

Given the power series An(x - 4)^n converges at x=7, what is its radius of convergence?

The radius of convergence R is at least the distance from the center 4 to 7, which is 3. Since it diverges at 9 (distance 5 from 4), the radius R is exactly 3.

How can we determine the interval of convergence for this power series?

The interval of convergence is centered at 4 with radius 3, so it extends from x=1 to x=7. Further analysis at the endpoints is required to confirm convergence at those points.

Why does the power series diverge at x=9?

Because x=9 is outside the radius of convergence (which is 3), the terms of the series do not tend to zero, leading to divergence.

What is the significance of the series converging at x=7 but not at x=9?

It indicates that the series has a finite radius of convergence of 3, with convergence occurring within that radius (up to x=7) and divergence beyond it (at x=9).

Can the series converge at the endpoint x=1?

Potentially, yes. Since x=1 is at a distance of 3 from 4, which matches the radius, convergence at the endpoint depends on the specific behavior of the series at that point and requires testing.

How do the convergence properties relate to the coefficients An in the series?

The convergence depends on the limit of the terms An(x-4)^n. The radius of convergence is determined by the growth rate of An, typically via the root or ratio test. Here, the convergence at x=7 and divergence at x=9 suggest the coefficients satisfy certain growth conditions related to the radius.

What methods can be used to find the radius of convergence in this context?

The root test or ratio test are commonly used. By analyzing the limit of |An|^{1/n} or |An+1 / An|, we can determine the radius R = 1 / limsup_{n→∞} |An|^{1/n}.

Additional Resources

If The Power Series An(x-4)^n Converges At X=7 And Diverges At X=9

In the realm of mathematical analysis, power series serve as fundamental tools for representing functions, approximating solutions, and exploring the behavior of functions within specific domains. When a power series converges at certain points but diverges at others, it reveals critical insights into the radius and interval of convergence, which are cornerstones in understanding the function's domain of validity. This article delves into a particular scenario: a power series of the form \(\sum_{n=0}^{\infty} A_n (x - 4)^n\) converging at \(x=7\) but diverging at \(x=9\). Such a case invites an in-depth analysis of the convergence properties, the underlying radius of convergence, and the implications for the function it represents.

Understanding Power Series and Their Convergence

What Is a Power Series?

A power series is an infinite sum expressed in the form:

\[
\sum_{n=0}^{\infty} A_n (x - c)^n
\]

where:

- \(A_n\) are coefficients that depend on \(n\),

- c is the center of the series, a fixed point around which the series is expanded,
- x is the variable.

Power series are foundational because they can represent a wide class of functions, such as exponential, sinusoidal, and polynomial functions, within their domain of convergence.

Radius and Interval of Convergence

The radius of convergence R determines the range of x around the center c for which the series converges. Specifically:

- The series converges absolutely for all x satisfying $|x - c| < R$,
- Diverges for $|x - c| > R$,
- The behavior at the boundary points $|x - c| = R$ requires separate analysis.

The interval of convergence is thus $[c - R, c + R]$, with potential inclusion or exclusion of the endpoints depending on the series.

Analyzing the Given Scenario: Convergence at $x=7$ and Divergence at $x=9$

Setting the Context: Centered at $x=4$

Given the power series:

$$
\sum_{n=0}^{\infty} A_n (x - 4)^n
$$

we can note that the series is centered at $c=4$. The points of interest are:

- $x = 7$,
- $x = 9$.

Calculating their distances from the center:

$$
|7 - 4| = 3
$$
$$
|9 - 4| = 5
$$

The key observations are:

- The series converges at $x=7$, which is 3 units from the center,
- The series diverges at $x=9$, which is 5 units from the center.

This immediately informs us about the radius of convergence.

Implication for the Radius of Convergence R

Since the series converges at $x=7$ but diverges at $x=9$, the radius of convergence must satisfy:

$$
R \geq 3,
$$
and simultaneously,

$$
R < 5,
$$

because the series diverges at $x=9$, which is 5 units away from the center.

Putting it together:

$$
3 \leq R < 5.
$$

This range suggests that the radius of convergence is somewhere between 3 and 5.

Mathematical Implications and Detailed Analysis

Determining the Radius of Convergence

The limsup (limit superior) of the nth root of the coefficients A_n governs the radius of convergence via the Cauchy-Hadamard formula:

$$
\frac{1}{R} = \limsup_{n \to \infty} |A_n|^{1/n}.
$$

From the given convergence/divergence at specific points:

- Since the series converges at $|x - 4| = 3$, the radius R must be at least 3.
- Since the series diverges at $|x - 4| = 5$, the radius R must be less than 5.

Thus:

$$
3 \leq R < 5.
$$

This is a crucial insight, as it narrows down the possible values of R.

Behavior at the Boundary Points

The behavior of the series at the boundary points $x = c \pm R$ depends on the nature of the coefficients A_n and the specific series. For example:

- If $x=7$, which is 3 units from the center, and the series converges, then at $|x - 4|=3$, the series is on or within the convergence boundary.
- At $x=9$, which is 5 units from the center, the series diverges, indicating the boundary at $|x - 4|=5$ is not included in the convergence interval.

Therefore, the interval of convergence is at least $[4 - R, 4 + R]$ with $3 \leq R < 5$, and does not include $|x - 4|=5$.

Deeper Insights: What Does This Mean for the Function?

Analytic Functions and Power Series

Power series typically represent analytic functions within their interval of convergence. The radius of convergence indicates the largest disk centered at c on which the function can be expressed as a power series.

In this context, the function represented by the series:

- Is analytic within $|x - 4| < R$,
- Has potential singularities or points of non-analyticity at $|x - 4| \geq R$.

Given the convergence at $x=7$ and divergence at $x=9$, the function is well-behaved (analytic) at least up to the boundary near $x=7$, but not beyond $x=9$.

Implications of the Divergence at $x=9$

The divergence at $x=9$ suggests that:

- There exists some singularity or discontinuity at or before $|x - 4|=5$.
- The series cannot be continued analytically beyond this radius.

This divergence also indicates that the function cannot be extended as an analytic function beyond the circle of radius $R < 5$.

Potential Scenarios for the Coefficients A_n

Given the convergence and divergence behavior:

- The coefficients A_n are likely to grow at a rate that ensures the series converges within $|x - 4| < R$ (with $R \geq 3$), but not beyond.
- The growth rate of A_n might be moderate to slow within the radius, but become unbounded as $n \to \infty$ if approaching the divergence boundary.

Broader Mathematical Context and Examples

Comparison with Known Power Series

To understand this scenario better, consider classical power series:

- Geometric Series: $\sum_{n=0}^\infty r^n$ converges for $|r| < 1$. The radius of convergence is 1.
- Exponential Series: $\sum_{n=0}^\infty \frac{x^n}{n!}$ converges for all x, i.e., $R = \infty$.

In our case, the power series behaves somewhat like a geometric series with a finite radius, but with boundary points that depend on the coefficients' growth.

Examples of Series with Similar Behavior

Suppose the coefficients A_n are such that:

- For $|x - 4| < 3$, the series converges,
- For $|x - 4| > 5$, it diverges,
- At $|x - 4|=3$, convergence is assured,

- At \(|x - 4|=5\), divergence occurs.

This could arise in series where the coefficients have a radius of convergence exactly between 3 and 5, perhaps due to a singularity located at a distance exactly 5 units from the center, or due to the growth rate of the coefficients.

Practical Implications and Applications

Mathematical Modeling and Approximation

Understanding the convergence behavior of such power series is vital in applications where functions are approximated using series expansions—such as in physics, engineering, and computational mathematics.

Knowing that the series converges at \(x=7\) but not at \(x=9\):

- Guides the choice of domain for accurate approximations.
- Indicates the necessity of alternative methods or series expansions

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