Solve The Exact Differential Equafion(1+exy+xexy)dx+(xex+2)dy=0

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Understanding how to solve differential equations is a fundamental aspect of advanced mathematics, especially in fields like physics, engineering, and applied sciences. One particular type of differential equation, known as an exact differential equation, offers a systematic approach to find solutions when specific conditions are met. In this article, we will delve into solving the exact differential equation:

$$
(1 + e^{xy} + x e^{xy})\, dx + (x e^{x} + 2)\, dy = 0
$$

This equation presents interesting features that require careful analysis. We will explore the steps involved in verifying its exactness, finding the potential function, and deriving the general solution. By understanding this process, you'll gain valuable insights into solving similar differential equations encountered in various scientific disciplines.

Understanding Exact Differential Equations

Before diving into solving the specific equation, it's essential to understand what makes a differential equation exact.

Definition of Exact Differential Equations

A first-order differential equation of the form:

$$
M(x, y) \, dx + N(x, y) \, dy = 0
$$

is called exact if there exists a function $\Psi(x, y)$ such that:

$$
\frac{\partial \Psi}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x, y)
$$

In this case, the original differential equation can be written as:

$$
d\Psi = 0
$$

which implies:

$$
\Psi(x, y) = C
$$

where C is an arbitrary constant.

Condition for Exactness

The key condition for the differential equation to be exact is:

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If this condition holds, the equation is exact, and we can proceed to find the potential function $\Psi(x, y)$.

Analyzing the Given Differential Equation

Let's analyze the specific differential equation:

$$
(1 + e^{xy} + x e^{xy})\, dx + (x e^{x} + 2)\, dy = 0
$$

Here, identify:

$$
M(x, y) = 1 + e^{xy} + x e^{xy}
$$
$$
N(x, y) = x e^{x} + 2
$$

Our goal is to verify whether this differential equation is exact.

Step 1: Check the Exactness Condition

Calculate:

\[
\frac{\partial M}{\partial y} \quad \text{and} \quad \frac{\partial N}{\partial x}
\]

Partial derivatives:

- \(\frac{\partial M}{\partial y}\):

Since \(M = 1 + e^{xy} + x e^{xy}\),

\[
\frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left(e^{xy} + x e^{xy} \right)
\]

- For \(e^{xy}\):

\[
\frac{\partial}{\partial y} e^{xy} = x e^{xy}
\]

- For \(x e^{xy}\):

\[
\frac{\partial}{\partial y} (x e^{xy}) = x^2 e^{xy}
\]

Putting it together:

\[
\frac{\partial M}{\partial y} = x e^{xy} + x^2 e^{xy} = e^{xy} (x + x^2)
\]

- \(\frac{\partial N}{\partial x}\):

Given \(N = x e^{x} + 2\),

\[
\frac{\partial N}{\partial x} = e^{x} + x e^{x} = e^{x} (1 + x)
\]

Compare the two derivatives:

\[
\frac{\partial M}{\partial y} = e^{xy} (x + x^2)
\]
\[

\frac{\partial N}{\partial x} = e^{x} (1 + x)
\]

Step 2: Determine if the Equation is Exact

To verify if the equation is exact, compare \(\frac{\partial M}{\partial y}\) and \(\frac{\partial N}{\partial x}\):

\[
e^{xy} (x + x^2) \quad \text{vs} \quad e^{x} (1 + x)
\]

Note that:

\[
e^{xy} = e^{x y}
\]
and

\[
e^{x} = e^{x}
\]

For the two derivatives to be equal:

\[
e^{xy} (x + x^2) = e^{x} (1 + x)
\]

Express \(e^{xy}\) in terms of \(e^{x}\) and \(e^{y}\):

\[
e^{xy} = e^{x y} = e^{x y}
\]

But unless \(x y = x\), these expressions are not generally equal. For the derivatives to be equal everywhere, the following must hold:

\[
e^{x y} (x + x^2) = e^{x} (1 + x)
\]

which simplifies to:

\[
e^{x y} (x + x^2) = e^{x} (1 + x)
\]

Dividing both sides by e^{x}:

$$
e^{x y - x} (x + x^2) = 1 + x
$$

This condition does not hold universally unless $x y - x = 0$, i.e., $x (y - 1) = 0$, which is only true along specific lines, not globally.

Conclusion: The given differential equation is not exact in its current form.

Step 3: Make the Equation Exact (Integrating Factor)

Since the equation is not exact, we can attempt to find an integrating factor to make it exact.

Common methods for integrating factors:

- Dependence on x only
- Dependence on y only
- Dependence on x and y

Given the form of the derivatives, examining the ratio:

$$
\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \quad \text{or} \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}
$$

can help identify an integrating factor.

Step 4: Applying an Integrating Factor

Given the complex nature, consider an integrating factor dependent on x or y.

Test for integrating factor depending on x:

Calculate:

$$
\mu(x) = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)
$$

or similar ratios. Alternatively, observe the structure of M and N for possible simplifications.

Step 5: Simplify and Find the Potential Function

Suppose we find an integrating factor $\mu(x, y)$ such that:

$$
\mu(x, y) M(x, y) \, dx + \mu(x, y) N(x, y) \, dy = 0
$$

becomes exact.

Given the complexity, one practical approach is to attempt substitution or transformation to reduce the equation to a more manageable form.

Alternative Approach: Substitution Method

Notice that the term e^{xy} suggests a substitution involving the product xy.

Step 1: Substitution $z = xy$

Let:

$$
z = xy
$$

then

$$
dz = y\, dx + x\, dy
$$

Rearranged:

$$
x\, dy = dz - y\, dx
$$

Our original differential equation:

\[

(1 + e^{z} + x e^{z})\, dx + (x e^{x} + 2)\, dy = 0
\]

Express \(dy\) in terms of \(dz\) and \(dx\):

\[
dy = \frac{dz - y\, dx}{x}
\]

Substituting into the equation:

\[
(1 + e^{z} + x e^{z})\, dx + (x e^{x} + 2) \left(\frac{dz - y\, dx}{x} \right) = 0
\]

Multiply through by \(x\):

\[
x (1 + e^{z} + x e^{z})\, dx + (x e^{x} + 2) (dz - y\, dx) = 0
\]

Distribute:

\[
x (1 + e^{z} + x e^{z})\, dx + (x e^{x} + 2) dz

Frequently Asked Questions

What is the key method to solve the differential equation (1 + e^{xy} + xe^{xy}) dx + (x e^x + 2) dy = 0?

The key method is to check if the equation is exact or can be made exact by finding an integrating factor, then proceed with integrating to find the solution.

Is the differential equation (1 + e^{xy} + xe^{xy}) dx + (x e^x + 2) dy = 0 exact, and how can we verify it?

To verify if the equation is exact, compute ∂/∂y of the coefficient of dx and ∂/∂x of the coefficient of dy; if they are equal, the equation is exact. In this case, they are not equal, so we need an integrating factor.

What substitution can simplify the differential equation (1 + e^{xy} + xe^{xy}) dx + (x e^x + 2) dy = 0?

A substitution like u = xy might help, as it simplifies the exponential terms, allowing the equation to be transformed into a more manageable form.

How do you find an integrating factor for this differential equation if it is not exact?

You can look for an integrating factor that depends on x or y alone. For example, dividing through by suitable functions or testing functions like $\mu(x)$ or $\mu(y)$ can help make the equation exact.

Can you provide the general solution approach for solving this differential equation?

First, verify if the equation is exact. If not, find an integrating factor. Then, integrate the relevant parts to find a potential function, and finally, express the general solution as an implicit relation between x and y.

What are common pitfalls when solving this kind of exact differential equation?

Common pitfalls include assuming the equation is exact without verification, overlooking the need for an integrating factor, and making algebraic errors during integration. Careful verification and systematic approach are essential.

Additional Resources

Exact Differential Equation

Solving differential equations is a cornerstone of advanced mathematics, especially in fields like physics, engineering, and applied sciences. Among various types, exact differential equations hold a special place because they often allow straightforward solutions once identified as exact. Today, we delve into a particularly interesting example:

$$
(1 + e^{xy} + x e^{xy})\, dx + (x e^{xy} + 2)\, dy = 0
$$

This equation is not only a rich test of differential equation solving techniques but also an excellent illustration of how to systematically approach and solve an exact differential equation.

Understanding the Form and Structure of the Differential Equation

Before jumping into the solution, it's essential to get a clear picture of the structure of the given differential equation.

Recognizing the Standard Form

A first-order differential equation can often be written as:

$$
M(x,y)\, dx + N(x,y)\, dy = 0
$$

In our case:

$$
M(x,y) = 1 + e^{xy} + x e^{xy}
$$

$$
N(x,y) = x e^{xy} + 2
$$

Our goal is to determine whether this differential equation is exact, meaning whether there exists a function $\Psi(x, y)$ such that:

$$
\frac{\partial \Psi}{\partial x} = M(x,y)
$$

$$
\frac{\partial \Psi}{\partial y} = N(x,y)
$$

If this condition holds, then solving the differential equation reduces to finding $\Psi(x, y)$.

Checking for Exactness

The Exactness Condition

A differential equation $M\, dx + N\, dy = 0$ is exact if:

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Let's compute these derivatives.

Calculating $\frac{\partial M}{\partial y}$

Given:

$$
M(x, y) = 1 + e^{xy} + x e^{xy}
$$

Differentiate with respect to y:

$$
\frac{\partial M}{\partial y} = 0 + \frac{\partial}{\partial y}(e^{xy}) + x \frac{\partial}{\partial y}(e^{xy})
$$

Note that:

$$
\frac{\partial}{\partial y}(e^{xy}) = e^{xy} \cdot x
$$

Therefore:

$$
\frac{\partial M}{\partial y} = x e^{xy} + x \cdot x e^{xy} = x e^{xy} + x^2 e^{xy} = e^{xy}(x + x^2)
$$

Calculating $\frac{\partial N}{\partial x}$

Given:

$$
N(x, y) = x e^{xy} + 2
$$

Differentiate with respect to x:

$$
\frac{\partial N}{\partial x} = e^{xy} + x \frac{\partial}{\partial x}(e^{xy})
$$

Since:

$$
\frac{\partial}{\partial x}(e^{xy}) = e^{xy} \cdot y
$$

we obtain:

$$
\frac{\partial N}{\partial x} = e^{xy} + x y e^{xy} = e^{xy}(1 + x y)
$$

Determining Exactness

Compare the derivatives:

$$
\frac{\partial M}{\partial y} = e^{xy}(x + x^2)
$$
$$
\frac{\partial N}{\partial x} = e^{xy}(1 + x y)
$$

Are these equal? Let's analyze:

$$
e^{xy}(x + x^2) \quad \text{vs} \quad e^{xy}(1 + x y)
$$

Dividing both sides by e^{xy} (which is always positive), the condition reduces to:

$$
x + x^2 = 1 + x y
$$

Rearranged:

$$
x^2 + x - 1 - x y = 0
$$

Or:

$$
x^2 + x - 1 = x y
$$

This is a relation involving x and y. Since the derivatives are not equal in general, the differential equation is not exact in its current form.

Transforming the Equation Into an Exact Form

Since the original differential equation isn't exact, we need to find an integrating factor $\mu(x, y)$ that makes it exact. Integrating factors are functions that, when multiplied by the entire equation, render it exact.

Common Strategies for Finding Integrating Factors

- Integrating factor depending solely on x
- Integrating factor depending solely on y
- More complex functions involving both x and y

Given the structure of M and N, a promising approach is to examine the form of the derivatives to identify a suitable integrating factor.

Searching for an Integrating Factor

Step 1: Examine $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$

Recall:

$$
\frac{\partial M}{\partial y} = e^{xy}(x + x^2)
$$
$$
\frac{\partial N}{\partial x} = e^{xy}(1 + x y)
$$

The difference:

$$
\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^{xy} [(x + x^2) - (1 + x y)]
$$

Simplify inside brackets:

$$
x + x^2 - 1 - x y
$$

Note that:

$$
x + x^2 - 1 - x y = (x^2 + x - 1) - x y
$$

Observe that the difference involves the term $x y$. This suggests exploring an integrating factor dependent on x or y to eliminate the $x y$ term.

Step 2: Considering an Integrating Factor $\mu(y)$

Suppose $\mu(y)$ depends only on y. Then, the modified M and N are:

$$
\tilde{M} = \mu(y) M

\]

$$
\tilde{N} = \mu(y) N
$$

The condition for exactness becomes:

$$
\frac{\partial (\mu M)}{\partial y} = \frac{\partial (\mu N)}{\partial x}
$$

Calculating:

$$
\frac{\partial (\mu M)}{\partial y} = \mu' (y) M + \mu(y) \frac{\partial M}{\partial y}
$$

$$
\frac{\partial (\mu N)}{\partial x} = \mu(y) \frac{\partial N}{\partial x}
$$

The exactness condition is:

$$
\mu' (y) M + \mu(y) \frac{\partial M}{\partial y} = \mu(y) \frac{\partial N}{\partial x}
$$

Dividing through by $\mu(y)$:

$$
\frac{\mu' (y)}{\mu(y)} M + \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Rearranged as:

$$
\frac{\mu' (y)}{\mu(y)} M = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}
$$

Recall:

$$
\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = e^{xy}(1 + x y) - e^{xy}(x + x^2) = e^{xy} [(1 + x y) - (x + x^2)]
$$

Simplify the brackets:

$$
1 + x y - x - x^2 = (1 - x) + x y - x^2
$$

Expressed as:

$$
(1 - x) + x y - x^2
$$

Substituting back, the relation becomes:

$$
\frac{\mu'(y)}{\mu(y)} M = e^{xy} [(1 - x) + x y - x^2]
$$

But since $M = 1 + e^{xy} + x e^{xy}$, this is complicated. Alternatively, perhaps a more straightforward approach is to look for an integrating factor depending only on x.

Choosing an Integrating Factor $\mu(x)$

Suppose $\mu(x)$ depends only on x. Then:

$$
\frac{\partial (\mu M)}{\partial y} = \frac{\partial (\mu N)}{\partial x}
$$

Similar to previous steps, this yields:

$$
\frac{\mu'(x)}{\mu(x)} M + \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Rearranged as:

\[
\

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