4. Solve The Homogeneous Differential Equation. 1 Point (xy)dx + Xdy = 0

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Solving differential equations is a fundamental skill in calculus, particularly when dealing with real-world problems involving rates, growth, and decay. Among these, homogeneous differential equations hold a special place due to their unique structure and solution methods. In this article, we will focus on solving the homogeneous differential equation expressed as (xy)dx + xdy = 0. This equation exemplifies the class of homogeneous equations, characterized by the ability to express the functions involved as homogeneous functions of the same degree. Understanding how to solve this equation will equip you with a valuable technique applicable to a broad range of similar problems.

Understanding Homogeneous Differential Equations

Before diving into the solution process, it is crucial to understand what makes a differential equation homogeneous and why such equations are important.

What Is a Homogeneous Differential Equation?

A differential equation is called homogeneous if it can be written in the form:

$$ \frac{dy}{dx} = \frac{F(x, y)}{G(x, y)} $$

where the function $F(x, y)$ and $G(x, y)$ are homogeneous functions of the same degree. Alternatively, in many cases, the differential equation can be expressed as:

$$ M(x, y) dx + N(x, y) dy = 0 $$

where both M and N are homogeneous functions of the same degree.

In our case, the equation:

$$ (xy) dx + x dy = 0 $$

can be identified as homogeneous because the terms involve products of x and y, both of which can be scaled uniformly.

Why Are Homogeneous Equations Important?

Homogeneous differential equations are significant because they often model real-world phenomena where variables are multiplicatively related, such as in physics, biology, and economics. Their structure allows for substitution methods that simplify the solving process, transforming the original complex equation into a more manageable form.

Method for Solving the Homogeneous Differential Equation (xy)dx + xdy = 0

Let’s now focus on solving the specific differential equation:

$$ (xy) dx + x dy = 0 $$

with the goal of finding the general solution.

Step 1: Rewrite the Equation

The first step involves rewriting the differential equation in a more workable form:

$$ (xy) dx + x dy = 0 $$

which can be written as:

$$ xy dx + x dy = 0 $$

Given that, divide through by x (assuming $x \neq 0$) to simplify:

$$ y dx + dy = 0 $$

or equivalently,

$$ y dx + dy = 0 $$

This form reveals a relation between dy and dx, which can be written as a differential equation:

$$ y dx + dy = 0 $$

or,

$$ dy = - y dx $$

This is a first-order linear differential equation in terms of y and x.

Step 2: Recognize the Type of Differential Equation

The simplified form:

$$ dy + y dx = 0 $$

or

$$ dy/dx + y = 0 $$

is a first-order linear differential equation. It is separable and straightforward to solve.

Step 3: Solve the Differential Equation

The differential equation:

$$ dy/dx + y = 0 $$

can be approached via separation of variables:

$$ \frac{dy}{dx} = - y $$

which separates as:

$$ \frac{dy}{y} = - dx $$

Integrate both sides:

$$ \int \frac{1}{y} dy = - \int dx $$

resulting in:

$$ \ln |y| = - x + C $$

where C is the constant of integration.

Exponentiating both sides:

$$ |y| = e^{-x + C} = e^{C} e^{-x} $$

Let $A = e^{C}$, which is an arbitrary positive constant, then:

$$ y = A e^{-x} $$

This is the general solution to the differential equation.

Alternative Approach: Substitution Method

While the previous steps directly lead to the solution, understanding the substitution method provides deeper insight into solving homogeneous equations.

Using the Substitution $v = \frac{y}{x}$

Since the original differential equation involves products of x and y, the substitution $v = y/x$ simplifies the problem:

$$ y = v x $$

Differentiating both sides with respect to x:

$$ dy/dx = v + x dv/dx $$

Substituting into the simplified differential equation:

$$ y dx + dy = 0 $$

becomes:

$$ v x dx + (v + x dv/dx) dx = 0 $$

or

$$ v x + v + x dv/dx = 0 $$

which simplifies to:

$$ v x + v + x dv/dx = 0 $$

Divide through by x:

$$ v + \frac{v}{x} + dv/dx = 0 $$

But note that the initial substitution leads to the direct solution, as shown earlier, confirming that the general solution is:

$$ y = A e^{-x} $$

Summary and Key Takeaways

- Homogeneous differential equations have a structure that allows for substitution methods, often simplifying the solving process.
- The specific equation $(xy) dx + x dy = 0$ can be simplified by dividing through by x, transforming it into a separable differential equation $dy/dx + y = 0$.
- Solving the separable equation yields the general solution:

$$ y = A e^{-x} $$

where A is an arbitrary constant.

- Alternatively, substitution $v = y/x$ confirms the solution and provides insight into the relation between variables.

Practical Applications of Solving Homogeneous Differential Equations

Homogeneous differential equations frequently appear in various fields:

- **Physics:** Modeling motion where variables are multiplicatively related, such as in cooling processes or harmonic oscillators.
- **Economics:** Describing proportional growth or decay, such as in investment models or population dynamics.
- **Biology:** Modeling enzyme kinetics or population interactions where rates depend on proportional factors.

Understanding how to solve these equations allows scientists and engineers to analyze complex systems and predict future behavior effectively.

Conclusion

Mastering the method to solve homogeneous differential equations like $(xy)\, dx + x\, dy = 0$ is an essential part of advanced calculus. By recognizing the structure, simplifying the equation, and applying appropriate methods such as separation of variables or substitution, you can find the general solution efficiently. This skill not only enhances your mathematical toolkit but also broadens your capacity to model and analyze real-world phenomena accurately. Whether in physics, economics, biology, or engineering, the ability to solve homogeneous differential equations is a powerful tool in your analytical arsenal.

Frequently Asked Questions

What is the general form of the homogeneous differential equation given as (xy)dx + xdy = 0?

The differential equation is homogeneous because it can be expressed as a function of the ratio y/x, and it can be written in the form M(x,y)dx + N(x,y)dy = 0 with M and N homogeneous functions of the same degree.

How do you verify that the differential equation (xy)dx + xdy = 0 is homogeneous?

By dividing both terms by x, the equation becomes y dx + dy = 0, which indicates the equation depends on the ratio y/x, confirming its homogeneity.

What substitution is typically used to solve a homogeneous differential equation like (xy)dx + xdy = 0?

The substitution y = vx (where v is a function of x) is used, which simplifies the equation by replacing y with vx and its derivative accordingly.

How do you solve the differential equation (xy)dx + xdy = 0 after substitution?

Substituting y = vx, differentiate to find dy/dx = v + x dv/dx, then substitute into the original equation to reduce it to a separable form and solve for v, then revert back to y.

What is the solution method for the differential equation (xy)dx + xdy = 0?

Convert it into a separable equation using substitution y = vx, solve the resulting algebraic and differential equations, and then substitute back to find y in terms of x.

Can you provide the step-by-step solution for (xy)dx + xdy = 0?

Yes. First, divide through by x: y dx + dy = 0. Then, rewrite as dy/dx = -y. Separating variables: dy/y = -dx. Integrate both sides: ln|y| = -x + C. Finally, exponentiate to find y: y = Ce^{-x}.

What is the general solution to the differential equation (xy)dx + xdy = 0?

The general solution is y = Ce^{-x}, where C is an arbitrary constant.

Are all homogeneous differential equations solvable using substitution methods like y = vx?

Most homogeneous equations can be solved using the substitution y = vx, especially when they can be expressed as functions of y/x, but some may require additional techniques depending on their form.

Additional Resources

Solve The Homogeneous Differential Equation. 1 Point (xy)dx + xdy = 0

Understanding how to solve differential equations is fundamental in calculus and its applications across physics, engineering, and other sciences. Among these, homogeneous differential equations are particularly important because they often appear in problems involving proportional relationships and scaled systems. The differential equation (xy)dx + xdy = 0 serves as a classic example, illustrating the methods used to tackle such equations. In this article, we will explore the nature of this differential equation, the techniques to solve it, and the insights it offers into the behavior of solutions.

Introduction to Homogeneous Differential Equations

Homogeneous differential equations are those where the equation can be expressed in terms of ratios involving the variables, typically allowing a substitution to simplify the problem. More formally, a differential equation is homogeneous if it can be written in the form:

\[\frac{dy}{dx} = F\left(\frac{y}{x}\right) \]

or equivalently, if the functions involved are homogeneous functions of the same degree.

Key features of homogeneous differential equations:

- They often involve terms where the variables are multiplied or divided together.
- They can be transformed into a separable form through substitution, simplifying the process of finding solutions.
- They are common in modeling phenomena with proportional relationships, such as growth rates and physics problems involving inverse-square laws.

Understanding the Differential Equation: (xy)dx + xdy = 0

Let's analyze the given differential equation:

\[(xy)dx + xdy = 0 \]

At first glance, this appears to be a first-order differential equation involving products of x and y. To understand how to solve it, we need to manipulate it into a more manageable form.

Step 1: Express in terms of dy/dx

Divide through by dx:

$$ (xy) + x \frac{dy}{dx} = 0 $$

which simplifies to:

$$ xy + x \frac{dy}{dx} = 0 $$

Now, solving for dy/dx:

$$ x \frac{dy}{dx} = -xy $$

$$ \frac{dy}{dx} = - y $$

This is a straightforward first-order differential equation, but notice that earlier steps involve some simplification. Alternatively, we can approach the original form more directly.

Step 2: Rewrite the original equation

Dividing the entire equation by x (assuming $x \neq 0$):

$$ \frac{xy}{x} dx + \frac{x}{x} dy = 0 $$

which simplifies to:

$$ y dx + dy = 0 $$

or:

$$ y dx + dy = 0 $$

This is a linear first-order differential equation in y and x, which can be manipulated into standard form:

$$ dy + y dx = 0 $$

Method of Solution: Using Substitution and Recognizing Homogeneity

The original differential equation can be approached using substitution techniques suitable for homogeneous equations.

Step 1: Recognize the structure

Observe that the original form involves products like xy, suggesting the substitution:

$$ v = \frac{y}{x} $$

which implies:

$$ y = v x $$

Differentiating both sides with respect to x:

$$ \frac{dy}{dx} = v + x \frac{dv}{dx} $$

Step 2: Rewrite the differential equation in terms of v

Recall that the original differential equation is:

$$ (xy) dx + x dy = 0 $$

Dividing through by x:

$$ y dx + dy = 0 $$

Substitute $y = v x$:

$$ v x dx + d(v x) = 0 $$

Differentiate $v x$:

$$ v x dx + (v dx + x dv) = 0 $$

which simplifies to:

$$ v x dx + v dx + x dv = 0 $$

Group terms:

$$ (v x + v) dx + x dv = 0 $$

Factor:

$$ v (x + 1) dx + x dv = 0 $$

Alternatively, noting the previous steps, perhaps a clearer path is to directly substitute into the original differential equation.

Alternative approach:

Start again from the original:

$$ (xy) dx + x dy = 0 $$

Divide through by x^2:

$$ y \frac{dx}{x} + dy/x = 0 $$

But perhaps a more straightforward method is to rearrange the original into a form suitable for substitution.

Simplified Approach: Using the substitution y = vx

Given:

$$ y = v x $$

then:

$$ dy/dx = v + x dv/dx $$

Substitute into the original differential equation:

$$ (x y) dx + x dy = 0 $$

which becomes:

$$ (x \cdot v x) dx + x (v + x dv/dx) dx = 0 $$

Simplify:

$$ v x^2 dx + x v dx + x^2 dv/dx dx = 0 $$

Divide through by $x^2 dx$:

$$ v + \frac{v}{x} + dv/dx = 0 $$

which simplifies to:

$$ dv/dx + \frac{v}{x} + v = 0 $$

or:

$$ dv/dx + v \left(1 + \frac{1}{x} \right) = 0 $$

This is a linear differential equation in v:

$$ dv/dx + P(x) v = 0 $$

where:

$$ P(x) = 1 + \frac{1}{x} $$

Solution of the Transformed Differential Equation

The differential equation:

$$ dv/dx + P(x) v = 0 $$

has the general solution:

$$ v(x) = C \exp \left(- \int P(x) dx \right) $$

Calculate the integral:

$$ \int P(x) dx = \int \left(1 + \frac{1}{x} \right) dx = \int 1 dx + \int \frac{1}{x} dx = x + \ln |x| + C $$

Ignoring the constant of integration for the integral (absorbing it into the exponential), the solution for v is:

$$ v(x) = C e^{ - (x + \ln |x|) } = C e^{-x} e^{ - \ln |x| } $$

Recall that:

$$ e^{ - \ln |x| } = |x|^{-1} $$

Thus:

$$ v(x) = C e^{-x} \frac{1}{|x|} $$

Since $y = v x$:

$$ y = v x = C e^{-x} \frac{1}{|x|} \times x $$

For $x > 0$, $|x| = x$, so:

$$ y = C e^{-x} $$

If $x < 0$, similar reasoning applies, but the absolute value remains, so the general solution is:

$$ y = C e^{-x} $$

where C is an arbitrary constant.

Final General Solution

The solution to the differential equation:

$$ (xy) dx + x dy = 0 $$

is:

$$ y = C e^{-x} $$

This exponential form reflects the decaying or growth behavior depending on the sign and value of the constant C.

Features, Pros, and Cons of the Method

Features:

- Utilizes substitution $y = v x$ to transform the original differential equation into a linear form.
- Exploits the homogeneous nature of the equation, simplifying the problem.
- Results in an explicit solution involving exponential functions.

Pros:

- Straightforward substitution makes solving homogeneous equations manageable.
- The method provides a general solution with arbitrary constant.
- Can be applied to a broad class of similar equations.

Cons:

- Requires recognition of the homogeneous structure and proper substitution.
- Can become complex if the substitution or algebraic manipulation is not straightforward.
- Assumes $x \neq 0$; special cases need separate handling.

Conclusion and Practical Implications

The differential equation (xy)dx + xdy = 0 exemplifies the elegant techniques required to solve homogeneous equations. By recognizing the structure and applying substitution, we reduce a complex differential equation into a simple linear form solvable through standard methods. The resulting exponential solution underscores the importance of transformations in differential equations, revealing the underlying behavior of the system modeled.

In practical terms, mastering such methods allows mathematicians, engineers, and scientists to analyze systems where proportional relationships dominate. Whether modeling heat transfer, population dynamics, or mechanical systems, the ability to solve homogeneous differential equations is a vital skill in analytical problem-solving.

Key takeaways:

- Homogeneous differential equations can often be tackled with substitution methods.
- Recognizing the structure of the equation simplifies the solution process.
- The solutions often involve exponential functions, indicating growth or decay behaviors.

In essence, the solution to (xy)dx + xdy = 0 not only provides a specific answer but also illustrates a broader technique applicable across various mathematical and real-world contexts.

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