Determine The Impulse Response Function For The Equation Y 6y 8y = G(t)

Determine The Impulse Response Function For The Equation Y 6y 8y = G(t)

When analyzing linear time-invariant (LTI) systems, understanding how the system responds to an impulsive input is crucial. The impulse response function, denoted as h(t), characterizes the system's behavior and allows us to predict its response to any arbitrary input via convolution. In this article, we will explore how to determine the impulse response function for the differential equation:

Y'' + 6y' + 8y = G(t)

(Note: The original equation appears to have a typographical error with "Y 6y 8y". Assuming the intended form is a second-order linear differential equation as shown above.)

Understanding the Differential Equation

Before delving into the calculation of the impulse response, it is essential to understand the structure and components of the differential equation in question.

Form of the Differential Equation

The standard form of a second-order linear differential equation with constant coefficients is:

Y'' + aY' + bY = G(t)

where:

- Y(t) is the output or response of the system.
- G(t) is the input or forcing function.
- a and b are constants representing damping or stiffness parameters.

In our case:

Y'' + 6Y' + 8Y = G(t)

Significance of the Impulse Response

The impulse response h(t) signifies the output of the system when G(t) is an impulse function δ(t):

Y(t) = h(t) when G(t) = δ(t)

Once h(t) is known, the response to any input G(t) can be obtained by convolution:

$Y(t) = (g h)(t) = \int_0^t G(\tau) h(t - \tau) d\tau$

Step-by-Step Process to Find the Impulse Response Function

Determining h(t) involves solving the differential equation with G(t) replaced by δ(t). The main steps include:

1. Formulate the homogeneous equation.
2. Calculate the system's characteristic equation.
3. Find the homogeneous solution.
4. Determine the particular solution (if necessary).
5. Apply initial conditions corresponding to the impulse input.
6. Express the solution as h(t).

Let's explore each step in detail.

1. Homogeneous Differential Equation

Set G(t) = 0 to analyze the system's natural response:

$Y'' + 6Y' + 8Y = 0$

2. Characteristic Equation

The characteristic equation derived from the homogeneous differential equation is:

$r^2 + 6r + 8 = 0$

Solve for r:

$r = [-6 \pm \sqrt{(36 - 32)}] / 2$

$r = [-6 \pm \sqrt{4}] / 2$

$r = [-6 \pm 2] / 2$

Thus, the roots are:

$r_1 = (-6 + 2) / 2 = -4$

$r_2 = (-6 - 2) / 2 = -4$

Note: Both roots are real and equal, r = -4.

3. Homogeneous Solution

Since the roots are repeated, the homogeneous solution is:

Y_h(t) = (A + Bt) e^{r t} = (A + Bt) e^{-4t}

where A and B are constants determined by initial conditions.

4. Initial Conditions for Impulse Response

The impulse response h(t) is a particular solution to the differential equation with the input δ(t). For systems described by the differential equation, the initial conditions are:

- y(0^-) = 0 (initial response before the impulse)
- y'(0^-) = 0 (initial velocity before the impulse)

At t = 0, the impulse δ(t) causes an instantaneous change in the system's velocity, which affects initial conditions as follows:

- y(0^+) = 0 (assuming zero initial displacement)
- y'(0^+) = 1 (due to the impulse's effect)

These jump conditions are derived from integrating the differential equation around t=0.

5. Applying the Initial Conditions

Using the properties of the Dirac delta function, the impulse causes the derivative of y(t) to jump by 1 at t=0:

- y(0^+) = 0
- y'(0^+) = 1

Applying these conditions to the general solution:

Y(t) = (A + Bt) e^{-4t}

At t=0:

Y(0) = A = y(0^+) = 0

Y'(t) = B e^{-4t} - 4(A + Bt) e^{-4t}

At $t=0$:

$Y'(0) = B - 4A = 1$

Since $A=0$:

$B - 0 = 1 \Rightarrow B=1$

6. Final Impulse Response Function

Substituting $A=0$ and $B=1$ into the homogeneous solution:

$h(t) = y(t) = (0 + 1\cdot t) e^{-4t} = t e^{-4t}$ for $t \geq 0$

This is the impulse response function of the system:

$h(t) = t e^{-4t}$ for $t \geq 0$

and $h(t) = 0$ for $t < 0$, consistent with causality.

Implications and Applications of the Impulse Response Function

The derived impulse response $h(t)$ provides valuable insights into the system's behavior:

- System Dynamics: The exponential decay e^{-4t} indicates damping, with the response diminishing over time.
- Predicting System Response: Using convolution, the system's output for any arbitrary input $G(t)$ can be calculated.
- Stability Analysis: The negative real roots suggest the system is stable, as responses decay to zero over time.
- Control Systems Design: Knowledge of $h(t)$ allows engineers to design controllers, filters, or compensators.

Additional Considerations and Advanced Topics

While the above process deals with the standard case where the input is an impulse, real-world applications often involve more complex inputs. Here are some related topics:

- **Laplace Transform Method:** An alternative approach to determine $h(t)$ involves transforming the differential equation into the s-domain, solving for $H(s)$, and then inverse transforming.

- **Effect of System Parameters:** Variations in coefficients a and b influence the natural response and damping characteristics.

- **Non-Impulse Inputs:** Once $h(t)$ is known, the response to any input $G(t)$ can be found through convolution.

- **Higher-Order Systems:** For systems of order higher than two, the process involves solving higher-degree characteristic equations and applying initial conditions accordingly.

Summary

Determining the impulse response function for the differential equation $Y'' + 6Y' + 8Y = G(t)$ involves solving the homogeneous equation with initial conditions derived from the properties of the Dirac delta function. The key steps include:

1. Formulating the homogeneous differential equation.
2. Solving for roots of the characteristic equation.
3. Applying initial conditions specific to the impulse.
4. Deriving the response function $h(t) = t e^{-4t}$ for $t \geq 0$.

This process not only offers insight into the system's dynamics but also provides a foundation for analyzing and designing systems in fields like electrical engineering, control systems, and signal processing.

Conclusion

Understanding how to determine the impulse response function is fundamental in analyzing linear systems. The derived response $h(t) = t e^{-4t}$ reflects how the system reacts immediately after an impulse input, highlighting the system's damping characteristics and stability. Mastery of this process enables engineers and scientists to model, predict, and optimize a wide range of dynamic systems effectively.

Keywords: Impulse Response Function, Differential Equation, System Response, Laplace Transform, Signal Processing, Control Systems, Homogeneous Solution, Causality, Stability

Frequently Asked Questions

What is the main goal when determining the impulse response function for the differential equation Y'' + 6Y' + 8Y = G(t)?

The main goal is to find the system's output response to a unit impulse input, which involves solving the homogeneous equation and then applying the convolution integral with G(t).

How do you find the homogeneous solution for the differential equation Y'' + 6Y' + 8Y = 0?

Solve the characteristic equation $r^2 + 6r + 8 = 0$, which factors as $(r + 2)(r + 4) = 0$, giving roots $r = -2$ and $r = -4$. The homogeneous solution is then $Y_h(t) = C1 e^{-2t} + C2 e^{-4t}$.

What is the significance of the impulse response function in the context of this differential equation?

The impulse response function characterizes how the system responds over time to a sudden impulse input, serving as the fundamental solution that can be used to find the response to any arbitrary input through convolution.

How do you incorporate the input function G(t) into the impulse response solution?

By convolving the impulse response function with G(t), typically using the integral $Y(t) = \int_0^t h(t - \tau) G(\tau) d\tau$, where h(t) is the impulse response.

What methods are used to find the impulse response function for this differential equation?

Methods include solving the homogeneous differential equation, finding the Green's function, and applying Laplace transforms to directly compute the impulse response.

How does the Laplace transform simplify the process of finding the impulse response function?

Applying Laplace transforms converts the differential equation into an algebraic equation, making it easier to solve for the transfer function and then inverse transforming to find the impulse response h(t).

What role do initial conditions play in determining the

impulse response function?

For the pure impulse response, initial conditions are typically zero, as the response is solely due to the impulse input; non-zero initial conditions modify the solution accordingly.

Can you write the general form of the impulse response function for the differential equation Y'' + 6Y' + 8Y = δ(t)?

Yes, the impulse response h(t) is given by the inverse Laplace transform of 1 / (s^2 + 6s + 8), which simplifies to h(t) = (1/2)(e^{-2t} - e^{-4t}) u(t), where u(t) is the unit step function.

What is the physical interpretation of the impulse response function in engineering systems?

It represents how a system reacts instantly to a brief input, revealing the system's inherent dynamics and stability characteristics, and serving as a building block for analyzing system responses to various inputs.

Additional Resources

Impulse Response Function

Understanding the Equation: Y'' + 6Y' + 8Y = G(t)

When delving into differential equations, especially those modeling physical systems, the concept of the impulse response function (IRF) is fundamental. It characterizes how a system reacts over time to an external input, specifically a sudden, short-lived stimulus such as a Dirac delta function.

The given differential equation:

$$
Y'' + 6Y' + 8Y = G(t)
$$

is a second-order linear nonhomogeneous differential equation, where:

- $Y(t)$ is the system’s output (dependent variable),
- $G(t)$ is an external forcing function (input),
- primes denote derivatives with respect to time t.

Our goal is to determine the impulse response function $h(t)$, which effectively describes the system's output $Y(t)$ when the input $G(t)$ is an impulse.

Breaking Down the Equation

The Homogeneous Part

The first step involves analyzing the homogeneous equation:

$$
Y'' + 6Y' + 8Y = 0
$$

This represents the system's natural response—how it behaves without external forcing.

The characteristic equation associated with this is:

$$
r^2 + 6r + 8 = 0
$$

Solving for r:

$$
r = \frac{-6 \pm \sqrt{36 - 32}}{2} = \frac{-6 \pm \sqrt{4}}{2}
$$

$$
r = \frac{-6 \pm 2}{2}
$$

Thus,

$$
r_1 = \frac{-6 + 2}{2} = -2
$$
$$
r_2 = \frac{-6 - 2}{2} = -4
$$

The homogeneous solution:

$$
Y_h(t) = C_1 e^{-2t} + C_2 e^{-4t}
$$

where C_1 and C_2 are constants determined by initial conditions.

Particular Solution and Forcing Function

When considering the impulse response, the external input $G(t)$ is modeled as a Dirac delta function $\delta(t)$, which is zero everywhere except at $t=0$, where it is infinitely high but with finite integral (area = 1).

The nonhomogeneous equation:

$$
Y'' + 6Y' + 8Y = \delta(t)
$$

describes an impulse applied at $t=0$. To find the IRF, we focus on the response of the system to this impulse, which is essentially the convolution of the impulse with the system's unit response.

Determining the Impulse Response Function $h(t)$

Step 1: Recognize the System as a Linear Time-Invariant System

The differential equation is linear and time-invariant (assuming constant coefficients). Its impulse response $h(t)$ is the system output when the input is a Dirac delta:

$$
h(t) = \text{Response of the system to } \delta(t)
$$

This response can be derived by solving the differential equation with initial conditions corresponding to an impulse.

Step 2: Applying Initial Conditions for the Impulse Response

The impulse causes an instantaneous change in the state variables. For a second-order differential equation, the standard initial conditions associated with an impulse are:

$$
Y(0^+) = 0
$$

$$
Y'(0^+) = 1

\]

This is because the impulse injects a unit of "velocity" at $t=0$, consistent with the properties of $\delta(t)$.

Step 3: Solving the Differential Equation with the Impulse

The solution involves:

1. Solving the homogeneous equation (already done).

2. Applying the initial conditions that incorporate the impulse effect.

The Laplace Transform method is particularly effective here because it simplifies handling the delta function:

$$
\mathcal{L}\{Y'' + 6Y' + 8Y\} = \mathcal{L}\{\delta(t)\}
$$

Recall:

$$
\mathcal{L}\{\delta(t)\} = 1
$$
$$
\mathcal{L}\{Y(t)\} = Y(s)
$$

Transforming the differential equation:

$$
s^2 Y(s) - s Y(0^-) - Y'(0^-) + 6 (s Y(s) - Y(0^-)) + 8 Y(s) = 1
$$

Assuming zero initial conditions before the impulse:

$$
Y(0^-) = 0, \quad Y'(0^-) = 0
$$

then:

$$
(s^2 + 6s + 8) Y(s) = 1
$$

Thus,

$$
Y(s) = \frac{1}{s^2 + 6s + 8}
$$

Factor the denominator:

$$
s^2 + 6s + 8 = (s + 2)(s + 4)
$$

So,

$$
Y(s) = \frac{1}{(s + 2)(s + 4)}
$$

Applying partial fraction decomposition:

$$
Y(s) = \frac{A}{s + 2} + \frac{B}{s + 4}
$$

Multiply both sides by $(s + 2)(s + 4)$:

$$
1 = A(s + 4) + B(s + 2)
$$

Set $s = -2$:

$$
1 = A(2) + B(0) \Rightarrow A = \frac{1}{2}
$$

Set $s = -4$:

$$
1 = A(0) + B(-2) \Rightarrow B = -\frac{1}{2}
$$

Thus,

$$
Y(s) = \frac{1/2}{s + 2} - \frac{1/2}{s + 4}
$$

Inverse Laplace Transform:

\[

h(t) = \frac{1}{2} e^{-2t} - \frac{1}{2} e^{-4t}
\]

for \(t \geq 0\).

Final Expression for the Impulse Response Function

\[