Find The Equation Of A Parabola With Focus (3, 4) And Directrix Y = 1.

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Understanding how to determine the equation of a parabola given its focus and directrix is a fundamental concept in conic sections, an essential topic in algebra and coordinate geometry. Whether you're a student preparing for exams, a teacher designing lessons, or an enthusiast exploring the properties of conic sections, mastering this process allows you to analyze and graph parabolas accurately. This comprehensive guide will walk you through the step-by-step procedure to find the equation of a parabola with the focus at (3, 4) and the directrix y = 1, covering key concepts, formulas, and detailed examples to enhance your understanding.

Understanding Parabolas: Focus and Directrix

Before diving into the calculations, it's crucial to understand the basic definition and properties of a parabola.

What is a Parabola?

A parabola is a symmetric, U-shaped curve that can be represented mathematically as the set of all points equidistant from a fixed point called the focus and a fixed line called the directrix.

Focus and Directrix: The Building Blocks

- Focus: A fixed point located inside the parabola. For our problem, the focus is at (3, 4).
- Directrix: A fixed line outside the parabola. For our case, the directrix is the line y = 1.

The parabola is the set of points (x, y) such that the distance to the focus equals the distance to the directrix.

Step-by-Step Guide to Find the Equation of the Parabola

Let's now systematically determine the equation of the parabola with the given focus and directrix.

Step 1: Identify the Focus and Directrix

- Focus: $F(3, 4)$
- Directrix: $y = 1$

Step 2: Understand the Geometric Relationship

Any point (x, y) on the parabola satisfies:

$$
\text{Distance to focus} = \text{Distance to directrix}
$$

Expressed mathematically:

$$
\sqrt{(x - 3)^2 + (y - 4)^2} = \text{Distance from } (x, y) \text{ to } y = 1
$$

Since the directrix is a horizontal line $y = 1$, the perpendicular distance from (x, y) to the directrix is:

$$
| y - 1 |
$$

The distance to the focus is:

$$
\sqrt{(x - 3)^2 + (y - 4)^2}
$$

Therefore, the defining equation of the parabola is:

$$
\sqrt{(x - 3)^2 + (y - 4)^2} = | y - 1 |
$$

Step 3: Remove the Absolute Value and Square Both Sides

Because the distance is always non-negative, and the right side is an absolute value, we consider:

$$
\sqrt{(x - 3)^2 + (y - 4)^2} = y - 1
$$

(Note: Since $y \geq 1$ for the parabola, the expression $y - 1$ is non-negative, so the equality holds as is.)

Squaring both sides:

$$
(x - 3)^2 + (y - 4)^2 = (y - 1)^2
$$

Step 4: Expand and Simplify

Expand both sides:

$$
(x - 3)^2 + (y - 4)^2 = (y - 1)^2
$$

$$
(x^2 - 6x + 9) + (y^2 - 8y + 16) = y^2 - 2y + 1
$$

Subtract y^2 from both sides:

$$
x^2 - 6x + 9 + y^2 - 8y + 16 = y^2 - 2y + 1
$$

$$
x^2 - 6x + 9 + y^2 - 8y + 16 - y^2 = - 2y + 1
$$

Simplify:

$$
x^2 - 6x + 25 - 8y = - 2y + 1
$$

Bring all terms to one side:

$$
x^2 - 6x + 25 - 8y + 2y - 1 = 0
$$

$$
x^2 - 6x + 24 - 6y = 0
$$

Step 5: Rewrite in Standard Parabola Form

Solve for y:

$$
x^2 - 6x + 24 = 6y
$$

$$
y = \frac{1}{6} x^2 - x + 4
$$

This is the equation of the parabola in vertex form:

$$
\boxed{
y = \frac{1}{6} x^2 - x + 4
}
$$

Key Features of the Derived Parabola

Understanding the properties of the parabola helps in graphing and analyzing its behavior.

Vertex of the Parabola

The vertex form of a quadratic:

$$
y = a x^2 + b x + c
$$

has its vertex at:

$$
x_v = - \frac{b}{2a}
$$

Calculating:

$$
a = \frac{1}{6}, \quad b = -1
$$

$$
x_v = - \frac{-1}{2 \times \frac{1}{6}} = \frac{1}{\frac{2}{6}} = \frac{1}{\frac{1}{3}} = 3
$$

Find y_v:

$$
y_v = \frac{1}{6} (3)^2 - (3) + 4 = \frac{1}{6} \times 9 - 3 + 4 = \frac{9}{6} - 3 + 4 = \frac{3}{2} - 3 + 4 = 0.5 - 3 + 4 = 1.5
$$

Vertex: $(3, 1.5)$

Axis of Symmetry

The parabola's axis of symmetry passes through the vertex:

$$
x = 3
$$

Focus and Directrix Confirmations

- The focus at (3, 4) lies above the vertex at (3, 1.5).
- The directrix $y = 1$ is below the vertex, consistent with the parabola opening upward.

Additional Insights and Applications

Understanding the equation derived above enables various applications:

Graphing the Parabola

- Plot the vertex at $(3, 1.5)$.
- Draw the axis of symmetry at $x = 3$.
- Use additional points to sketch the parabola accurately, such as calculating y for $x = 2$ and $x = 4$.

Real-World Examples

- Parabolas model projectile motion, satellite dishes, and car headlights.
- Knowing the focus and directrix allows engineers and scientists to design paraboloid structures with precise focus points.

Mathematical Significance

- This process demonstrates the direct geometric relationship between focus, directrix, and the parabola's equation.
- It exemplifies how algebraic manipulation reflects geometric properties.

Summary of the Process

To recap, here is a concise step-by-step summary:

1. Identify focus and directrix.
2. Set the distance from a generic point to focus equal to the distance to the directrix.

3. Express the distances algebraically and set the equations equal.
4. Square both sides to eliminate radicals and absolute values.
5. Simplify the resulting equation.
6. Express the parabola in standard quadratic form.
7. Determine key features such as vertex, axis of symmetry, and direction.

Conclusion

Finding the equation of a parabola given its focus and directrix is a fundamental skill that combines geometric intuition with algebraic proficiency. In this case, with focus (3, 4) and directrix $y = 1$, we derived the parabola's equation:

$$
\boxed{
y = \frac{1}{6} x^2 - x + 4
}
$$

This equation encapsulates the parabola's shape, position, and key properties, providing a foundation for graphing, analysis, and real-world applications. Mastering this method enhances your understanding of conic sections and strengthens your overall mathematical problem-solving skills.

Keywords for SEO Optimization:
- Equation of a parabola
- Focus and directrix of a parabola
- How to find

Frequently Asked Questions

What is the general method to find the equation of a parabola given its focus and directrix?

To find the equation, identify the vertex as the midpoint between the focus and the directrix, determine the parabola's orientation, and then use the definition that the parabola consists of points equidistant from the focus and the directrix to derive its equation.

Given the focus at (3, 4) and the directrix y = 1, how do you find the vertex of the parabola?

The vertex lies midway between the focus and the directrix along the axis of symmetry. Since the focus is at $y=4$ and the directrix is $y=1$, the vertex's y-coordinate is $(4 + 1)/2 = 2$. The x-coordinate is the same as the focus, so the vertex is at (3, 2).

What is the axis of symmetry for the parabola with focus (3, 4) and directrix y = 1?

The axis of symmetry is the vertical line passing through the focus and the vertex, which in this case is $x = 3$.

How do you derive the equation of the parabola with focus (3, 4) and directrix y = 1?

Using the definition of a parabola, for any point (x, y) on the parabola, the distance to the focus (3, 4) equals its perpendicular distance to the directrix $y=1$. This leads to the equation: sqrt[(x - 3)^2 + (y - 4)^2] = |y - 1|.

How can the equation be simplified to standard form for the parabola with focus (3, 4) and directrix y = 1?

Square both sides to eliminate the square root: (x - 3)^2 + (y - 4)^2 = (y - 1)^2. Expand and simplify to obtain the standard quadratic form of the parabola.

What is the explicit equation of the parabola with focus at (3, 4) and directrix y=1?

Starting from (x - 3)^2 + (y - 4)^2 = (y - 1)^2, simplifying yields (x - 3)^2 + (y - 4)^2 = y^2 - 2y + 1. Further simplification results in the parabola's equation: (x - 3)^2 = 4(y - 2).

What is the significance of the equation (x - 3)^2 = 4(y - 2) in this context?

This is the standard form of the parabola with vertex at (3, 2) that opens upward, with focus at (3, 4) and directrix $y=1$, confirming the parabola's shape and position.

Additional Resources

Find the Equation of a Parabola with Focus (3, 4) and Directrix y = 1

Understanding the precise mathematical representation of conic sections, particularly parabolas, is fundamental in geometry, physics, engineering, and numerous applied sciences. When given a focus point and a directrix, the challenge lies in deriving the unique quadratic equation that encapsulates this geometric relationship. In this detailed exploration, we will dissect the process of finding the equation of a parabola with focus at (3, 4) and directrix $y = 1$, providing a clear, methodical approach that combines geometric intuition with algebraic rigor.

Introduction to Parabolas: Geometric Foundations

A parabola is a conic section that can be defined as the locus of points equidistant from a fixed point called the focus and a fixed line called the directrix. This geometric property is the cornerstone of understanding the parabola's shape and equation.

Key Definitions:

- Focus: A fixed point outside the parabola. For our problem, it is at (3, 4).
- Directrix: A fixed line outside the parabola. Here, it is $y = 1$.
- Locus of Points: The set of all points (x, y) such that the distance to the focus equals the perpendicular distance to the directrix.

The defining property of a parabola makes it unique among conic sections, characterized by its symmetric, "U"-shaped curve.

Geometric Intuition of the Focus and Directrix

Understanding the positions of the focus and directrix is crucial for deriving the parabola's equation.

- Focus at (3, 4): This point lies above the line $y = 1$, indicating the parabola opens upward.
- Directrix $y = 1$: A horizontal line below the focus, serving as a boundary from which the parabola is defined.

Since the focus and directrix are both aligned vertically, the parabola's axis of symmetry is vertical, passing through the focus and perpendicular to the directrix.

Step-by-Step Derivation of the Equation

Deriving the parabola's equation involves translating the geometric definition into an algebraic form.

1. Establish the Parabola's Basic Properties

Given the focus F = (3, 4) and directrix y = 1:

- The parabola's vertex is midway between the focus and the directrix along the axis of symmetry.
- The axis of symmetry is the vertical line x = 3 because the focus and directrix are vertically aligned.

2. Find the Vertex Coordinates

- The focus's y-coordinate: 4
- The directrix y-coordinate: 1
- The vertex lies halfway between these two along the axis of symmetry:

$$
\text{Vertex } (h, k) = \left(3, \frac{4 + 1}{2} \right) = (3, 2.5)
$$

3. Determine the Distance from Vertex to Focus (p)

- The distance from the vertex to the focus along the axis of symmetry is:

$$
p = \text{distance from } (3, 2.5) \text{ to } (3, 4) = 4 - 2.5 = 1.5
$$

- Since the parabola opens upward (focus above vertex), the value of p is positive: $p = 1.5$.

4. Write the Standard Equation of the Parabola

For a parabola with a vertical axis of symmetry and vertex at (h, k), the general form is:

\[

(y - k)^2 = 4p(x - h)
\]

or

\[
(y - k)^2 = 4p(x - h)
\]

But since the parabola opens upward, the standard form becomes:

\[
(y - k)^2 = 4p(x - h)
\]

Alternatively, for a parabola opening upward/downward with axis of symmetry along y, the equation is:

\[
(x - h)^2 = 4p(y - k)
\]

In our case, because the axis is vertical (x = 3), the standard form is:

\[
(x - 3)^2 = 4p(y - 2.5)
\]

5. Substitute the Value of p

\[
(x - 3)^2 = 4 \times 1.5 \times (y - 2.5) = 6(y - 2.5)
\]

Thus, the equation of the parabola is:

\[
\boxed{
(x - 3)^2 = 6(y - 2.5)
}
\]

Verification of the Equation

To ensure correctness, verify that the focus point (3, 4) satisfies the derived equation:

$$
(x - 3)^2 = 6(y - 2.5)
$$

Plugging in (3, 4):

$$
(3 - 3)^2 = 6(4 - 2.5) \Rightarrow 0 = 6 \times 1.5 \Rightarrow 0 = 9
$$

This suggests a discrepancy; the point (3, 4) does not satisfy the equation as written. The issue stems from the form used. Our earlier assumption of the standard equation needs to be revisited.

Correction:

Since the parabola opens upward with focus at (3, 4), the standard form for a parabola opening upward is:

$$
(x - h)^2 = 4p(y - k)
$$

with vertex at (h, k), and focus at (h, k + p).

Given that:

- Focus at (3, 4)
- Vertex at (3, 2.5)

then:

$$
p = \text{distance from vertex to focus} = 4 - 2.5 = 1.5
$$

and the focus's y-coordinate:

$$
k + p = 4 \Rightarrow 2.5 + p = 4 \Rightarrow p = 1.5
$$

\]

which matches our previous calculation.

Now, the standard form:

$$
(x - h)^2 = 4p(y - k)
$$

becomes:

$$
(x - 3)^2 = 4 \times 1.5 \times (y - 2.5) = 6(y - 2.5)
$$

Now, check the focus point:

$$
x = 3, \quad y = 4
$$

Plug in:

$$
(3 - 3)^2 = 6(4 - 2.5) \Rightarrow 0 = 6 \times 1.5 \Rightarrow 0 = 9
$$

which is inconsistent, indicating that the focus point does not lie on the parabola defined by this equation. This suggests a need for a more accurate approach considering the directrix.

Alternative Approach: Using the Distance Formula

Given the geometric definition, the parabola consists of points equidistant from the focus and the directrix:

$$
\text{Distance to focus} = \text{Distance to directrix}
$$

1. Express the Distance to the Focus

For an arbitrary point (x, y):

\[
D_f = \sqrt{(x - 3)^2 + (y - 4)^2}
\]

2. Express the Distance to the Directrix

The directrix is y = 1. The perpendicular distance from (x, y) to y = 1 is:

\[
D_d = | y - 1 |
\]

3. Set the Distances Equal

\[
\sqrt{(x - 3)^2 + (y - 4)^2} = | y - 1 |
\]

4. Square Both Sides to Eliminate the Square Root

\[
(x - 3)^2 + (y - 4)^2 = (y - 1)^2
\]

5. Expand Both Sides

\[
(x - 3)^2 + y^2 - 8 y + 16 = y^2 - 2 y + 1
\]

Subtract \(y^2 \) from both sides:

\[
(x - 3)^2 - 8 y + 16 = -2 y + 1
\]

Bring all to one side:

\[

$$
(x - 3)^2 - 8 y + 16 + 2 y - 1 = 0
$$

Simplify:

$$
(x - 3)^2 - 6 y + 15 = 0
$$

6. Solve for y

$$
6 y = (x - 3)^2 + 15
$$

$$
y = \frac{(x - 3)^2 + 15}{6}
$$

This represents the explicit equation of the parabola.

Final Equation and Its Interpretation

The derived equation:

$$
\boxed{
y = \frac{(x - 3)^2 + 15}{6}
}
$$

describes the parabola with focus at (3, 4) and directrix y = 1.

Key observations:

- The parabola opens upward, consistent with the focus being

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