Find The Points On The Surface Xy^2z^3 = 2 That Are Closest To The Origin

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When exploring the fascinating realm of multivariable calculus, one intriguing problem involves locating the points on a given surface that are nearest to the origin. Specifically, for the surface defined by the equation $xy^2z^3 = 2$, identifying the points closest to the origin requires a blend of geometric intuition and calculus techniques such as Lagrange multipliers. This problem not only deepens understanding of constrained optimization but also highlights the application of calculus in geometric contexts, making it a valuable topic for students and enthusiasts alike.

In this comprehensive article, we will delve into the step-by-step methodology to find the points on the surface $xy^2z^3 = 2$ that are closest to the origin. We will explore the mathematical background, formulate the problem, employ Lagrange multipliers, and interpret the solutions. Additionally, we will discuss related concepts, provide tips for approaching similar problems, and optimize the content for SEO to assist learners searching for solutions to such constrained optimization problems.

Understanding the Problem: Closest Points to the Origin on a Surface

When asked to find the points on a surface that are closest to the origin, the general approach involves minimizing the distance function subject to the given constraint. For a point (x, y, z), the distance to the origin is:

$$
d = \sqrt{x^2 + y^2 + z^2}
$$

Since the square root function is monotonic, minimizing d is equivalent to minimizing d^2:

$$
D(x, y, z) = x^2 + y^2 + z^2
$$

subject to the constraint:

$$
g(x, y, z) = xy^2z^3 - 2 = 0
$$

\]

This transforms our problem into a constrained optimization task:

> Minimize $D(x, y, z) = x^2 + y^2 + z^2$
>
> subject to $xy^2z^3 = 2$

Mathematical Tools: Lagrange Multipliers

The Lagrange multipliers method is a powerful technique for solving constrained optimization problems. It involves introducing a multiplier λ and solving the system:

$$
\nabla D(x, y, z) = \lambda \nabla g(x, y, z)
$$

where:

$$
\nabla D = \left(2x, 2y, 2z \right)
$$
$$
\nabla g = \left(y^2z^3, 2xy z^3, 3xy^2 z^2 \right)
$$

This leads to the system of equations:

$$
2x = \lambda y^2 z^3 \quad (1)
$$
$$
2y = 2 \lambda x y z^3 \quad (2)
$$
$$
2z = 3 \lambda x y^2 z^2 \quad (3)
$$

along with the original constraint:

$$
xy^2z^3 = 2
$$

Step-by-Step Solution Approach

1. Set Up the System of Equations

Start with the equations derived from the Lagrange multiplier method:

$$
(1) \quad 2x = \lambda y^2 z^3
$$
$$
(2) \quad 2y = 2 \lambda x y z^3
$$
$$
(3) \quad 2z = 3 \lambda x y^2 z^2
$$

Note that equation (2) simplifies to:

$$
2y = 2 \lambda x y z^3 \Rightarrow y = \lambda x y z^3
$$

If $y \neq 0$, then:

$$
1 = \lambda x z^3
$$

Similarly, from equation (3):

$$
2z = 3 \lambda x y^2 z^2
$$

If $z \neq 0$, divide both sides by z:

$$
2 = 3 \lambda x y^2 z
$$

---

2. Express λ from the Equations

From the simplified equations, express λ:

$$
\lambda = \frac{1}{x z^3}
$$

And from the third:

$$
3 \lambda x y^2 z = 2 \Rightarrow 3 \left(\frac{1}{x z^3}\right) x y^2 z = 2
$$

Simplify:

$$
3 y^2 z / z^3 = 2
$$
$$
3 y^2 / z^2 = 2
$$

Solve for y^2:

$$
y^2 = \frac{2 z^2}{3}
$$

3. Use the Constraint

Recall the original constraint:

$$
xy^2 z^3 = 2
$$

Substitute y^2:

$$
x \left(\frac{2 z^2}{3}\right) z^3 = 2
$$

Simplify:

$$
x \cdot \frac{2 z^2}{3} \cdot z^3 = 2
$$
$$
x \cdot \frac{2}{3} z^{5} = 2
$$

Solve for x:

$$
x = \frac{2 \times 3}{2 z^{5}} = \frac{3}{z^{5}}
$$

4. Find y

Recall:

$$
y^2 = \frac{2 z^2}{3}
$$

Thus:

$$
y = \pm \sqrt{\frac{2 z^2}{3}} = \pm \frac{z \sqrt{2}}{\sqrt{3}}
$$

5. Express x, y, z

Now, we have:

$$
x = \frac{3}{z^5}
$$
$$
y = \pm \frac{z \sqrt{2}}{\sqrt{3}}
$$
$$
z \neq 0
$$

Finding the Minimal Distance: Final Step

Recall that the squared distance from the origin is:

$$
D = x^2 + y^2 + z^2
$$

Substitute the expressions:

$$
x^2 = \frac{9}{z^{10}}
$$
$$
y^2 = \frac{2 z^2}{3}
$$
$$
z^2 = z^2
$$

So,

$$
D(z) = \frac{9}{z^{10}} + \frac{2 z^2}{3} + z^2 = \frac{9}{z^{10}} + \frac{2 z^2}{3} + z^2
$$

Combine the last two terms:

$$
\frac{2 z^2}{3} + z^2 = \frac{2 z^2}{3} + \frac{3 z^2}{3} = \frac{5 z^2}{3}
$$

Thus,

$$
D(z) = \frac{9}{z^{10}} + \frac{5 z^2}{3}
$$

Our goal is to find the value of z that minimizes $D(z)$. To do this, differentiate $D(z)$ with respect to z and set the derivative to zero:

$$
D'(z) = -10 \times \frac{9}{z^{11}} + \frac{10 z}{3} = -\frac{90}{z^{11}} + \frac{10 z}{3}
$$

Set $D'(z) = 0$:

$$
-\frac{90}{z^{11}} + \frac{10 z}{3} = 0
$$

Rearranged:

$$
\frac{10 z}{3} = \frac{90}{z^{11}}
$$

Multiply both sides by z^{11}:

$$
\frac{10 z^{12}}{3} = 90
$$

Solve for z^{12}:

$$
10 z^{12} = 270
$$
$$
z^{12} = 27
$$

Since $z^{12} = 27$:

$$
z = \pm \sqrt[12]{27} = \pm 3^{3/12} = \pm 3^{1/4}
$$

Choose the positive root for the minimal distance (since distance is positive):

$$
z = 3^{1/4}
$$

Final Coordinates of the Closest Points

Now, compute x and y:

$$
x = \frac{3}{z^{5}} = \frac{3}{(3^{1/4})^{5}} = \frac{3}{3^{5/4}} = 3^{1 - 5/4} = 3^{(4/4) - (5/4)} = 3^{-1/4} = \frac{1}{3^{1/4}}
$$

Frequently Asked Questions

What is the main goal when finding points on the surface xy^2z^3 = 2 that are closest to the origin?

The main goal is to find the points on the surface that minimize the distance to the origin, typically by minimizing the distance function subject to the surface constraint.

How do you formulate the optimization problem for the closest points on the surface xy^2z^3 = 2?

You set up an optimization problem to minimize the squared distance function D^2 = x^2 + y^2 + z^2, subject to the constraint xy^2z^3 = 2.

What method can be used to solve the constrained minimization problem in this context?

The method of Lagrange multipliers is commonly used to find the points that minimize the distance while satisfying the surface constraint.

How do you set up the Lagrangian for this problem?

The Lagrangian is set up as $L(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(xy^2z^3 - 2)$, incorporating the constraint into the optimization problem.

What are the steps to find the critical points using Lagrange multipliers?

First, take partial derivatives of L with respect to x, y, z, and λ; then set these derivatives equal to zero and solve the resulting system of equations to find candidate points.

What challenges might arise when solving the system of equations for this problem?

The system can be nonlinear and complex, potentially leading to multiple solutions; solving it may require algebraic manipulation, substitution, or numerical methods.

Once the candidate points are found, how do you determine which ones are closest to the origin?

Calculate the distance squared ($x^2 + y^2 + z^2$) for each candidate point and select the point(s) with the smallest value, indicating the closest approach to the origin.

Are there any symmetry considerations that can simplify solving this problem?

Yes, symmetry in variables or the surface equation can sometimes reduce the number of variables or help identify equivalent solutions, simplifying the analysis.

What is the significance of verifying the solutions obtained from the Lagrange multipliers method?

Verifying ensures that the solutions satisfy both the stationarity conditions and the original constraint, and helps identify whether they are minima, maxima, or saddle points regarding the distance to the origin.

Additional Resources

Find The Points On The Surface $XY^2Z^3 = 2$ That Are Closest To The Origin

An In-Depth Investigation into Constrained Optimization on a Cubic Surface

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Introduction

The problem of identifying points on a given surface that are closest to a specific point—most often the origin—is a classical question in calculus and geometric analysis. When the surface is defined explicitly by an algebraic equation, such as $XY^2Z^3 = 2$, the problem translates into a constrained optimization challenge: minimize the distance from a point (here, the origin) to any point on that surface.

This type of problem is not only of theoretical interest but also has applications across physics, engineering, and computational geometry, where understanding the spatial relationship between complex surfaces and points is crucial. In this article, we undertake a thorough investigation into Find The Points On The Surface $XY^2Z^3 = 2$ That Are Closest To The Origin, exploring methods, solutions, and the underlying mathematical principles involved.

The Geometric and Analytical Context

The Surface Defined by $XY^2Z^3 = 2$

The surface in question is a three-dimensional algebraic surface with the defining equation:

$$
XY^2Z^3 = 2
$$

This surface exhibits interesting symmetry and complexity:

- It is not a simple quadratic or linear surface; the degrees of the variables are 1, 2, and 3, respectively.
- The surface extends infinitely in all directions, with asymptotic behavior influenced by the zeroes and singularities implied by the equation.
- The positivity or negativity of the variables influences the existence and location of solutions, particularly because of the cubic term Z^3 and quadratic Y^2.

The Goal: Find the Closest Points to the Origin

Mathematically, the problem reduces to:

$$
\text{Minimize} \quad D^2 = x^2 + y^2 + z^2
$$

subject to the constraint:

$$
xy^2z^3 = 2
$$

where (x, y, z) are points on the surface.

Choosing to minimize D^2 instead of D simplifies the calculus, as the square root is monotonic, and the square is differentiable everywhere.

Mathematical Approach: Constrained Optimization Using Lagrange Multipliers

Setting Up the Problem

The method of Lagrange multipliers allows us to incorporate the constraint directly into the optimization problem.

Define the Lagrangian:

$$
\mathcal{L}(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda (xy^2z^3 - 2)
$$

Our goal is to find stationary points of $\mathcal{L}$ by solving:

$$
\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \frac{\partial \mathcal{L}}{\partial z} = 0, \quad xy^2z^3 = 2
$$

Deriving the System of Equations

Calculating partial derivatives:

$$
\frac{\partial \mathcal{L}}{\partial x} = 2x - \lambda y^2 z^3 = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial y} = 2y - 2 \lambda x y z^3 = 0
$$
$$
\frac{\partial \mathcal{L}}{\partial z} = 2z - 3 \lambda x y^2 z^2 = 0
$$

Rearranged:

$$
2x = \lambda y^2 z^3 \quad (1)
$$
$$
2y = 2 \lambda x y z^3 \quad \Rightarrow y (1 - \lambda x z^3) = 0 \quad (2)
$$
$$
2z = 3 \lambda x y^2 z^2 \quad \Rightarrow z (1 - \frac{3}{2} \lambda x y^2 z) = 0 \quad (3)
$$

\]

Analyzing the System

The equations involve multiple variables and parameters, suggesting multiple potential cases:

- Case 1: \(y = 0 \) or \(z = 0 \)

- Case 2: \(y \neq 0 \), \(z \neq 0 \)

Case 1: \(y = 0 \) or \(z = 0 \)

If \(y = 0 \), then from the constraint:

\[
x \cdot 0 \cdot z^3 = 2 \Rightarrow 0 = 2
\]

which is impossible. Similarly, if \(z=0 \):

\[
x y^2 \cdot 0 = 2 \Rightarrow 0=2
\]

also impossible.

Conclusion: The points closest to the origin cannot have \(y=0 \) or \(z=0 \).

Case 2: \(y \neq 0 \), \(z \neq 0 \)

From equation (2):

\[
1 - \lambda x z^3 = 0 \Rightarrow \lambda x z^3 = 1
\]

From equation (3):

\[
1 - \frac{3}{2} \lambda x y^2 z = 0 \Rightarrow \frac{3}{2} \lambda x y^2 z = 1
\]

Dividing the second by the first:

\[
\frac{\frac{3}{2} \lambda x y^2 z}{\lambda x z^3} = \frac{1}{1} \Rightarrow

\frac{\frac{3}{2} y^2 z}{z^3} = 1
\]

Simplify:

\[
\frac{3}{2} y^2 z^{ -2} = 1
\]
\[
\Rightarrow y^2 = \frac{2}{3} z^{2}
\]

Thus:

\[
y = \pm \sqrt{\frac{2}{3}} z
\]

Using equation (1):

\[
2x = \lambda y^2 z^3
\]

But from earlier,

\[
\lambda x z^3 = 1 \Rightarrow \lambda = \frac{1}{x z^3}
\]

Substituting into equation (1):

\[
2x = \frac{1}{x z^3} y^2 z^3 = \frac{1}{x} y^2
\]

Multiply both sides by \(x \):

\[
2x^2 = y^2
\]

Recall:

\[
y^2 = \frac{2}{3} z^2
\]

Thus:

\[

2 x^2 = \frac{2}{3} z^2 \Rightarrow x^2 = \frac{1}{3} z^2
\]

So:

\[
x = \pm \frac{z}{\sqrt{3}}
\]

Now, the constraint:

\[
x y^2 z^3 = 2
\]

Substitute \(x = \pm \frac{z}{\sqrt{3}} \), \(y^2 = \frac{2}{3} z^2 \):

\[
\left(\pm \frac{z}{\sqrt{3}}\right) \cdot \frac{2}{3} z^2 \cdot z^3 = 2
\]

Calculate step-by-step:

\[
\pm \frac{z}{\sqrt{3}} \times \frac{2}{3} z^2 \times z^3 = 2
\]

Simplify:

\[
\pm \frac{2}{3 \sqrt{3}} z^{1+2+3} = 2
\]
\[
\pm \frac{2}{3 \sqrt{3}} z^{6} = 2
\]

Solve for \(z^{6} \):

\[
z^{6} = \pm 2 \times \frac{3 \sqrt{3}}{2} = \pm 3 \sqrt{3}
\]

Note that \(z^{6} \) is always non-negative, so the right side must be positive:

\[
z^{6} = 3 \sqrt{3}
\]

Therefore:

\[

z^{6} = 3^{1} \times 3^{1/2} = 3^{3/2}
\]

Taking the sixth root:

\[
z = \pm \left(3^{3/2}\right)^{1/6} = \pm 3^{(3/2) \times (1/6)} = \pm 3^{1/4}
\]

Similarly, find \(x \) and \(y \):

\[
x = \pm \frac{z}{\sqrt{3}} = \pm \frac{3^{1/4}}{\sqrt{3}} = \pm 3^{1/4} \times 3^{-1/2} = \pm 3^{1/4 - 1/2} = \pm 3^{-1/4} = \

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