Find The Relative Maximum And Minimum Values. F(x,y)=x^(2) Y^(2)-2x 12y-15

Find The Relative Maximum And Minimum Values. F(x,y)=x^{2} y^{2}-2x + 12y - 15

Understanding how to find the relative maximum and minimum values of a function of multiple variables is a fundamental concept in multivariable calculus. These points, known as local extrema, are crucial in various fields such as engineering, economics, and physical sciences because they identify the peaks and valleys within a function's domain. In this article, we will explore the process of finding the relative maximum and minimum values of the function:

$$ F(x,y) = x^{2} y^{2} - 2x + 12y - 15 $$

through detailed steps, including calculating critical points, analyzing the second derivatives, and interpreting the results.

Understanding the Function F(x,y)

Before diving into calculus techniques, let's understand the structure of the given function:

- Term $x^{2} y^{2}$: A quadratic term in both variables, forming a sort of "bowl" shape that depends on both x and y.
- Linear terms $-2x$ and $12y$: These influence the position of the function's critical points.
- Constant -15: A vertical shift that affects the overall value but does not influence the shape or the location of extrema.

The goal is to find points (x,y) where the function attains local maxima or minima, which entails analyzing the behavior of $F(x,y)$ through calculus.

Step 1: Find Critical Points

Critical points occur where the first-order partial derivatives of F with respect to x and y are zero or undefined. Since $F(x,y)$ is polynomial, the derivatives are defined everywhere, so critical points are found by solving:

$$
\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0
$$

Calculating Partial Derivatives

- Partial derivative with respect to x:

$$
\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(x^{2} y^{2} - 2x + 12y - 15) = 2x y^{2} - 2
$$

- Partial derivative with respect to y:

$$
\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}(x^{2} y^{2} - 2x + 12y - 15) = 2 x^{2} y + 12
$$

Setting Partial Derivatives Equal to Zero

To find critical points:

$$
2 x y^{2} - 2 = 0 \quad \Rightarrow \quad 2 x y^{2} = 2 \quad \Rightarrow \quad x y^{2} = 1
$$

$$
2 x^{2} y + 12 = 0 \quad \Rightarrow \quad 2 x^{2} y = -12 \quad \Rightarrow \quad x^{2} y = -6
$$

Step 2: Solve the System of Equations

We now have the system:

$$
\begin{cases}
x y^{2} = 1 \\
x^{2} y = -6
\end{cases}
$$

Our goal is to solve for x and y.

Express x in terms of y

From the first equation:

$$
x y^{2} = 1 \quad \Rightarrow \quad x = \frac{1}{y^{2}}
$$

Substitute into the second equation

Plugging $x = \frac{1}{y^{2}}$ into the second:

$$
\left(\frac{1}{y^{2}}\right)^{2} y = -6
$$

Simplify:

$$
\frac{1}{y^{4}} \cdot y = -6
$$

$$
\frac{y}{y^{4}} = -6 \quad \Rightarrow \quad \frac{1}{y^{3}} = -6
$$

$$
\Rightarrow y^{3} = -\frac{1}{6}
$$

Find y and x

- Solution for y:

$$
y = \sqrt[3]{-\frac{1}{6}} = - \sqrt[3]{\frac{1}{6}} = - \frac{1}{\sqrt[3]{6}}
$$

- Solution for x:

$$
x = \frac{1}{y^{2}} = \frac{1}{\left(- \frac{1}{\sqrt[3]{6}} \right)^2} = \frac{1}{\frac{1}{\left(\sqrt[3]{6}\right)^2}} = \left(\sqrt[3]{6}\right)^2
$$

Thus, the critical point is:

$$
\boxed{
\left(x, y \right) = \left(\left(\sqrt[3]{6} \right)^2, - \frac{1}{\sqrt[3]{6}} \right)
}
$$

Step 3: Classify Critical Points Using Second Derivative Test

To determine whether this critical point is a maximum, minimum, or saddle point, we employ the second derivative test for functions of two variables.

Calculate Second-Order Partial Derivatives

- $\frac{\partial^{2} F}{\partial x^{2}}$:

$$
\frac{\partial}{\partial x}\left(2 x y^{2} \right) = 2 y^{2}
$$

- $\frac{\partial^{2} F}{\partial y^{2}}$:

$$
\frac{\partial}{\partial y}\left(2 x^{2} y + 12 \right) = 2 x^{2}
$$

- Mixed partial derivatives:

$$
\frac{\partial^{2} F}{\partial x \partial y} = \frac{\partial}{\partial y}\left(2 x y^{2} - 2 \right) = 4 x y
$$

By Clairaut's theorem, the mixed partial derivatives are equal:

$$
\frac{\partial^{2} F}{\partial y \partial x} = 4 x y
$$

Compute the Discriminant D

The second derivative test involves calculating:

$$
D = \left(\frac{\partial^{2} F}{\partial x^{2}} \right) \left(\frac{\partial^{2} F}{\partial y^{2}} \right) - \left(\frac{\partial^{2} F}{\partial x \partial y} \right)^2
$$

At the critical point (x, y):

$$
D = (2 y^{2})(2 x^{2}) - (4 x y)^2 = 4 x^{2} y^{2} - 16 x^{2} y^{2} = -12 x^{2} y^{2}
$$

Substituting the critical point values:

$$
x = (\sqrt[3]{6})^2, \quad y = - \frac{1}{\sqrt[3]{6}}
$$

Compute x^{2}:

$$
x^{2} = \left((\sqrt[3]{6})^{2} \right)^2 = (\sqrt[3]{6})^{4}
$$

Compute y^{2}:

$$
y^{2} = \left(- \frac{1}{\sqrt[3]{6}} \right)^2 = \frac{1}{(\sqrt[3]{6})^{2}}
$$

Now, multiply:

$$
x^{2} y^{2} = (\sqrt[3]{6})^{4} \times \frac{1}{(\sqrt[3]{6})^{2}} = (\sqrt[3]{6})^{4 - 2} = (\sqrt[3]{6})^{2}
$$

Recall that $\sqrt[3]{6} = 6^{1/3}$, so:

$$
x^{2} y^{2} = (6^{1/3})^{2} = 6^{2/3}
$$

Thus,

$$
D = -12 \times 6^{2/3}
$$

\]

Since $6^{2/3} > 0$, the discriminant $D < 0$, indicating that the critical point is a saddle point.

Summary of Results

Critical Point	Classification	Explanation
$\left((\sqrt[3]{6})^2, - \frac{1}{\sqrt[3]{6}} \right)$	Saddle Point	Because the discriminant $D < 0$

Frequently Asked Questions

How do I find the relative maxima and minima of the function $F(x, y) = x^2y^2 - 2x + 12y - 15$?

To find the relative extrema, first compute the partial derivatives of F with respect to x and y, set them equal to zero to find critical points, then use the second derivative test to classify each point as a maximum, minimum, or saddle point.

What are the steps involved in solving for critical points of $F(x, y) = x^2y^2 - 2x + 12y - 15$?

The steps include calculating the first partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$, setting them equal to zero, solving the resulting system of equations for x and y to identify critical points.

How do I apply the second derivative test to classify the critical points of the function?

Calculate the second partial derivatives F_{xx}, F_{yy}, and F_{xy} at each critical point. Then compute the discriminant $D = F_{xx} F_{yy} - (F_{xy})^2$. If $D > 0$ and $F_{xx} > 0$, it's a local minimum; if $D > 0$ and $F_{xx} < 0$, it's a local maximum; if $D < 0$, the point is a saddle point.

Can the critical points of $F(x, y) = x^2y^2 - 2x + 12y - 15$ be saddle points?

Yes, if the second derivative test yields a discriminant $D < 0$ at a critical point, that point is a saddle point, indicating neither a maximum nor a minimum.

What is the significance of finding relative maxima and minima in multivariable functions like F(x, y)?

Finding these points helps identify the highest and lowest values of the function locally, which is essential in optimization problems, economics, engineering, and understanding the behavior of the surface defined by the function.

Are there any critical points for $F(x, y) = x^2y^2 - 2x + 12y - 15$ that are neither maxima nor minima?

Yes, critical points where the second derivative test indicates a saddle point are neither maxima nor minima. These points are saddle points where the surface curves upward in one direction and downward in another.

How does the presence of the term x^2y^2 affect the nature of the critical points?

The x^2y^2 term introduces a quadratic interaction between x and y, which can create multiple critical points and saddle points, making the classification more complex compared to simpler functions.

Can I use a graphing calculator or software to find and classify the extrema of F(x, y)?

Yes, tools like Desmos, GeoGebra, WolframAlpha, or graphing calculators can help visualize the function, find critical points numerically, and assist in applying the second derivative test for classification.

Additional Resources

Find The Relative Maximum And Minimum Values is a fundamental concept in multivariable calculus, essential for understanding the behavior of functions of several variables. In this article, we'll explore how to determine the relative extrema of the function $F(x, y) = x^2 y^2 - 2x + 12y - 15$. This process involves critical point analysis, the use of partial derivatives, and the second derivative test for functions of two variables. Through a detailed step-by-step approach, we'll uncover the points where the function attains relative maxima or minima, analyze their nature, and discuss the implications of these findings.

Understanding the Function $F(x, y)$

Before diving into the process of finding relative extrema, it’s important to understand the structure and characteristics of the function:

$$F(x, y) = x^2 y^2 - 2x + 12y - 15$$

This function is a polynomial in two variables, combining quadratic and linear terms. The interaction between $x^2 y^2$ and the linear parts creates a complex surface with potential peaks and valleys—hence the importance of identifying its relative maximum and minimum points.

Features of the function:
- It contains a mixed quadratic term $x^2 y^2$, which is always non-negative and grows rapidly as $|x|$ or $|y|$ increases.
- Linear terms $-2x$ and $+12y$ influence the shape and position of potential extrema.
- The constant term -15 shifts the entire surface vertically.

Finding Critical Points

The first step in locating relative maxima and minima is to find the critical points, where the gradient of F equals zero.

Calculating Partial Derivatives

Compute the first-order partial derivatives:

$$
\frac{\partial F}{\partial x} = 2x y^2 - 2
$$
$$
\frac{\partial F}{\partial y} = 2x^2 y + 12
$$

Critical points occur where both derivatives are zero:

$$
2x y^2 - 2 = 0 \quad \Rightarrow \quad 2x y^2 = 2 \quad \Rightarrow \quad x y^2 = 1
$$
$$
2x^2 y + 12 = 0 \quad \Rightarrow \quad 2x^2 y = -12 \quad \Rightarrow \quad x^2 y = -6
$$

Solving the System of Equations

From the first equation:

$$
x y^2 = 1 \quad \Rightarrow \quad x = \frac{1}{y^2}
$$

\]

Substitute into the second equation:

\[
x^2 y = -6
\]
\[
\left(\frac{1}{y^2}\right)^2 y = -6
\]
\[
\frac{1}{y^4} y = -6
\]
\[
\frac{1}{y^3} = -6
\]
\[
y^3 = -\frac{1}{6}
\]
\[
y = -\left(\frac{1}{6}\right)^{1/3}
\]

Since \(y^3 = -\frac{1}{6} \), the real cube root is:

\[
y \approx -\left(\frac{1}{6}\right)^{1/3}
\]

Numerically, \(\left(\frac{1}{6}\right)^{1/3} \approx 0.551 \), so:

\[
y \approx -0.551
\]

Now, substitute back to find \(x \):

\[
x = \frac{1}{y^2} = \frac{1}{(0.551)^2} \approx \frac{1}{0.303} \approx 3.30
\]

Thus, the critical point is approximately:

\[
\boxed{
\left(x, y \right) \approx (3.30, -0.551)
}
\]

Classifying Critical Points Using the Second Derivative Test

Once critical points are identified, the next step is to determine whether these points are local maxima, minima, or saddle points. The second derivative test for functions of two variables involves computing the second derivatives and evaluating the discriminant.

Second Partial Derivatives

Calculate the second derivatives:

$$
F_{xx} = \frac{\partial^2 F}{\partial x^2} = 2 y^2
$$
$$
F_{yy} = \frac{\partial^2 F}{\partial y^2} = 2 x^2
$$
$$
F_{xy} = \frac{\partial^2 F}{\partial x \partial y} = 4 x y
$$

Evaluate these at the critical point $(3.30, -0.551)$:

$$
F_{xx} \approx 2 \times (-0.551)^2 = 2 \times 0.303 \approx 0.606
$$
$$
F_{yy} \approx 2 \times (3.30)^2 = 2 \times 10.89 \approx 21.78
$$
$$
F_{xy} \approx 4 \times 3.30 \times (-0.551) \approx 4 \times (-1.814) \approx -7.256
$$

Discriminant Calculation

The discriminant D is:

$$
D = F_{xx} F_{yy} - (F_{xy})^2
$$
$$
D \approx 0.606 \times 21.78 - (-7.256)^2
$$
$$
D \approx 13.2 - 52.66 \approx -39.46
$$

Since $D < 0$, the critical point is a saddle point, not a maximum or minimum.

Implications of the Analysis

The analysis indicates that the only critical point found is a saddle point. Thus, the function does not have a local maximum or minimum at this point. To further understand the behavior of $F(x, y)$:

- The surface exhibits saddle-like behavior near $(3.30, -0.551)$.
- The function's quadratic term $x^2 y^2$ dominates for large $|x|$ and $|y|$, leading to unbounded growth, which suggests no absolute maximum or minimum exists globally.
- The linear terms shift the surface and influence the position of saddle points but do not create local extrema in this particular case.

Summary of Key Steps and Findings

- Critical points are found where the gradient vector components are zero, leading to the approximate critical point $(3.30, -0.551)$.
- The second derivative test reveals this critical point is a saddle point, indicating neither a local maximum nor a minimum.
- The function's nature suggests it is unbounded above and below, with the quadratic term dominating for large values of $|x|$ and $|y|$.

Pros and Cons of the Analytical Approach

Pros:
- Systematic method for finding and classifying extrema.
- Provides insight into the surface's local behavior.
- Applicable to a wide range of functions.

Cons:
- Computationally intensive for more complex functions.
- Approximate numerical solutions may be necessary when algebraic solutions are intractable.
- Does not directly address global extrema, only local.

Conclusion

The process of finding the relative maximum and minimum values of $F(x, y) = x^2 y^2 - 2x + 12y - 15$ highlights the importance of critical point analysis and the second derivative test in multivariable calculus. Although the critical point identified is a saddle point, understanding these features provides valuable insights into the function's behavior, especially in applications involving optimization and modeling. Recognizing the limitations, such as the unbounded nature of the function, underscores the importance of comprehensive analysis when dealing with functions of multiple variables. This approach, combining derivatives and classification tests, remains a fundamental tool for mathematicians and engineers alike in the quest to optimize complex surfaces.

Find The Relative Maximum And Minimum Values F X Y X 2 Y 2 2x 12y 15

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