Find The Volume Of The Solid Whose Base Is Bounded By The Circle X^2 Y^2=4

Find The Volume Of The Solid Whose Base Is Bounded By The Circle X^2 + Y^2 = 4

Introduction

Understanding how to compute the volume of a solid bounded by a specific base and a given surface is a fundamental aspect of multivariable calculus. In this article, we will explore the problem of finding the volume of a solid whose base is the circle defined by the equation $x^2 + y^2 = 4$. We will examine the methods involved, primarily focusing on double integrals in polar coordinates, which simplify calculations involving circular regions. We will also discuss the types of surfaces that could be relevant and how to set up the integral for the volume calculation.

Understanding the Geometric Context

The Base: The Circle $x^2 + y^2 = 4$

The base of the solid is a circle centered at the origin with radius 2. Its equation is:

$$
x^2 + y^2 = 4
$$

This circle is symmetric about both axes and covers all points (x, y) such that the distance from the origin is less than or equal to 2.

The Solid: Variations and Surfaces

To determine the volume of the solid, we need to specify the surface that forms the upper boundary of the solid. Common options include:

- A plane, e.g., $z = k$, where k is a constant.
- A paraboloid, e.g., $z = x^2 + y^2$.
- A more general surface, such as $z = f(x, y)$.

In many standard problems, the surface is given explicitly or implied. For illustration, assume the surface is defined by a function $z = f(x, y)$. The volume V then becomes:

$$
V = \iint_{D} f(x, y) \, dA
$$

where D is the domain of the base — the circle $x^2 + y^2 \leq 4$.

Setting Up the Integral for Volume Calculation

Choosing Coordinates: Cartesian vs. Polar

Calculating the volume involves integrating over the domain D. Given that the base is circular, polar coordinates are often more convenient:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

with the Jacobian determinant:

$$
dA = r \, dr \, d\theta
$$

The circle $x^2 + y^2 = 4$ in polar coordinates becomes:

$$
0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi
$$

Formulating the Double Integral

Suppose the surface is $z = f(x, y)$. The volume is:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} f(r, \theta) \, r \, dr \, d\theta
$$

If the surface is a simple function such as $z = k$, the integral simplifies to:

$$
V = k \times \text{Area of the base} = k \times \pi \times 2^2 = 4\pi k
$$

But for more complex surfaces, the integral involves the specific form of f.

Example: Volume Under a Paraboloid Surface

Defining the Surface

Let's consider a specific example where the surface is a paraboloid:

$$
z = 4 - (x^2 + y^2)
$$

This surface opens downward, reaching zero at the boundary $x^2 + y^2 = 4$.

Visualizing the Solid

The paraboloid intersects the plane $z=0$ at the boundary of the base circle, creating a solid that is bounded above by the paraboloid and below by the xy-plane (or the base circle).

Setting Up and Computing the Volume

The volume under $z = 4 - r^2$ over the domain $r \in [0, 2]$, $\theta \in [0, 2\pi]$, is:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} [4 - r^2] \, r \, dr \, d\theta
$$

Calculating the integral:

$$
V = \int_{0}^{2\pi} d\theta \int_{0}^{2} (4r - r^3) dr
$$

First, the inner integral:

$$
\int_{0}^{2} 4r \, dr = 2 \times 4 \times \frac{r^2}{2} \bigg|_{0}^{2} = 4 \times 2^2 = 4 \times 4 = 16
$$

$$
\int_{0}^{2} r^3 \, dr = \frac{r^4}{4} \bigg|_{0}^{2} = \frac{16}{4} = 4
$$

Thus,

$$
V = \int_{0}^{2\pi} (16 - 4) \, d\theta = \int_{0}^{2\pi} 12 \, d\theta = 12 \times 2\pi = 24\pi
$$

Hence, the volume of the solid under the paraboloid above the circle is 24π cubic units.

General Approach to Find the Volume

Step 1: Define the Surface $z = f(x, y)$

Identify the surface that bounds the solid above the base. The surface could be specified explicitly or inferred from the problem context.

Step 2: Describe the Domain D

In this case, the domain is the circle $x^2 + y^2 \leq 4$. Using polar coordinates simplifies the limits:

$$
0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi
$$

Step 3: Set Up the Double Integral

Express the volume as:

$$
V = \iint_D f(x, y) \, dA
$$

which in polar coordinates becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} f(r, \theta) \, r \, dr \, d\theta
$$

Step 4: Perform the Integration

Calculate the inner integral with respect to r, then the outer integral with respect to θ. Pay attention to the form of $f(r, \theta)$ to evaluate the integral effectively.

Applications and Variations

Different Surfaces

Depending on the problem, the surface could be:

- A plane, such as $z = c$, where c is constant.
- A paraboloid, $z = a - b(x^2 + y^2)$.
- An arbitrary surface, $z = f(x, y)$.

Each case involves similar steps but different integrand functions.

Other Bases

While here the base is a circle, similar methods apply to other bounded regions, such as:

- Ellipses
- Rectangles
- Irregular shapes (using appropriate coordinate transformations)

Additional Techniques and Tips

- **Coordinate Transformation:** Use polar coordinates for circular bases to simplify limits and integrals.
- **Symmetry:** Exploit symmetry of the region and surface to reduce computation time.
- **Integral Evaluation:** Be familiar with standard integrals involving powers of r and trigonometric functions.
- **Software Tools:** Use computational tools like WolframAlpha, MATLAB, or Wolfram Mathematica for complex integrals.

Conclusion

Calculating the volume of a solid with a circular base involves understanding the geometry of the base, choosing appropriate coordinate systems, and setting up the correct double integral of the bounding surface over the domain. Whether the surface is a simple plane, paraboloid, or an arbitrary function, the approach remains consistent: parametrize the domain, express the surface in terms of the chosen coordinates, and evaluate the resulting integral. Mastery of these techniques is essential for solving a wide range of problems in multivariable calculus and geometric analysis.

References & Further Reading

- Stewart, James. Calculus: Early Transcendentals. Chapters on Multiple Integrals.
- Thomas' Calculus, 13th Edition.
- Online resources such as Khan Academy's Multivariable Calculus section.
- Wolfram MathWorld: Double Integrals and Coordinate Systems.

This comprehensive overview provides the foundation for solving volume problems involving circular bases and various surfaces,

Frequently Asked Questions

What is the base shape of the solid when bounded by the circle $x^2 + y^2 = 4$?

The base shape of the solid is a circle with radius 2, as given by the equation $x^2 + y^2 = 4$.

How do you set up the integral to find the volume of the solid with this base?

You can set up a double integral over the circular region using either Cartesian or polar coordinates, integrating the height function if given, or assuming a specific surface.

What is the typical height function used if the problem involves a specific surface?

If the problem specifies a surface, such as $z = f(x, y)$, then the volume is found by integrating $f(x, y)$ over the base region: $V = \iint_D f(x, y)\, dA$.

How can the symmetry of the circle $x^2 + y^2 = 4$ simplify the volume calculation?

The symmetry allows converting to polar coordinates, simplifying the limits of integration and calculations due to the circular symmetry.

What are the limits of integration in polar coordinates for this circle?

In polar coordinates, r varies from 0 to 2, and θ varies from 0 to 2π.

If the solid is bounded above by z = 0 and below by some surface, how do you find the volume?

You set up a double integral over the base region, integrating the height function (z) or the difference between the surfaces, depending on the problem's specifics.

Can you give an example of a specific problem involving finding the volume with this base?

Yes. For instance, find the volume of the solid bounded above by $z = 4 - x^2 - y^2$ over the base circle $x^2 + y^2 \leq 4$. The volume is obtained by integrating z over the circle.

What is the formula for the volume of a solid with a circular base of radius 2 and height given by a function z = f(x, y)?

$V = \iint_D f(x, y)\, dA$, where D is the disk $x^2 + y^2 \leq 4$. In polar coordinates, $V = \int_0^{2\pi} \int_0^{2} f(r, \theta)\, r\, dr\, d\theta$.

What common mistakes should be avoided when calculating the volume of a solid with a circular base?

Avoid mixing coordinate systems improperly, neglecting the Jacobian when converting to polar coordinates, and forgetting to set correct limits for the integration region.

Additional Resources

Find The Volume Of The Solid Whose Base Is Bounded By The Circle $X^2 + Y^2 = 4$

Understanding how to determine the volume of a solid with a specific base is a fundamental aspect of calculus, particularly in the realm of multivariable calculus. In this article, we will thoroughly explore the problem of finding the volume of a solid whose base is bounded by the circle $x^2 + y^2 = 4$. This particular problem not only illustrates the application of double integrals but also offers insight into the geometric interpretation of volume calculation in a two-dimensional region extended into the third dimension through a specified surface or height function.

Introduction to the Problem

When asked to find the volume of a solid with a given base, the key is to understand the shape and boundaries of the base region, then determine the surface height function over that region. In this case, the base is bounded by the circle $x^2 + y^2 = 4$, which is a circle centered at the origin with radius 2. The problem typically involves defining a height function $z = f(x,y)$ over this region and then integrating that function over the bounded area.

The general approach to such problems involves:

- Describing the base region mathematically.
- Choosing an appropriate coordinate system (Cartesian, polar, etc.).
- Setting up the double integral based on the height function.
- Computing the integral to find the volume.

Understanding the Base Region: The Circle $x^2 + y^2 = 4$

Geometric Properties of the Circle

The circle $x^2 + y^2 = 4$ has:

- Center at the origin (0,0).
- Radius $r = 2$.

This circle divides the plane into two parts: the interior (the disk $x^2 + y^2 \leq 4$) and the exterior. Since the problem specifies that the base is bounded by this circle, the region of integration is the disk:

$$
D = \{ (x,y) \mid x^2 + y^2 \leq 4 \}
$$

Understanding the symmetry and shape of this region is crucial because it affects how we set up the integral.

Advantages of Symmetry

- The circle is symmetric about both axes, simplifying calculations.
- The region is a perfect disk, which lends itself well to conversion into polar coordinates.

Choosing the Coordinate System

While Cartesian coordinates are straightforward, converting the problem into polar coordinates simplifies the integration process significantly, especially given the circular boundary.

Polar Coordinates Overview

- Transformation equations:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

- Jacobian for the transformation:

$$
dx\,dy = r \, dr\, d\theta
$$

- Region D in polar coordinates:

$$
0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi
$$

This transformation converts the circular boundary into a simple radial boundary, making the double integral setup more straightforward.

Defining the Height Function $z = f(x,y)$

The problem statement usually involves a specified surface or height function that extends vertically over the base region. Common functions include:

- $z = 1$, which would create a cylindrical solid.
- $z = x$ or $z = y$, which creates a skewed solid.
- $z = \text{constant}$, for a uniform height.

For the purpose of this comprehensive review, we will analyze the volume of the solid when the height function is:

Case 1: $z = 1$ (a constant height)

This models a simple cylinder with the base as the disk $x^2 + y^2 \leq 4$.

Case 2: $z = f(x,y) = y$

This introduces a more interesting surface, creating a "slope" over the disk.

Case 3: $z = \sqrt{4 - x^2 - y^2}$

This models a hemisphere of radius 2, which is a common surface in volume problems.

In this review, we will primarily analyze the case where $z = 1$, then briefly discuss how other height functions would modify the approach.

Setting Up the Double Integral

For $z = 1$: The Volume is Simply the Area of the Base Region

Because the height is constant ($z = 1$), the volume V is:

$$
V = \iint_D 1 \, dx\, dy
$$

which simplifies to computing the area of the disk D:

$$
V = \text{Area of } D = \pi r^2 = \pi \times 2^2 = 4\pi
$$

Pros:
- Straightforward calculation.
- Demonstrates the direct relation between area and volume in uniform height cases.

Cons:
- Limited to constant height functions; does not account for more complex surfaces.

For General Height Function $z = f(x,y)$: The Volume Integral

The volume becomes:

$$
V = \iint_D f(x,y) \, dx\, dy
$$

or, in polar coordinates:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} f(r, \theta) \, r\, dr\, d\theta
$$

This setup allows for flexible computation based on the chosen surface.

Calculating the Volume: Step-by-Step Approach

Step 1: Convert the Region to Polar Coordinates

- r ranges from 0 to 2.
- θ ranges from 0 to 2π.

Step 2: Express the Height Function in Polar Coordinates

- For constant $z=1$:

$$
f(r, \theta) = 1
$$

- For other functions, express x and y in terms of r and θ.

Step 3: Set Up the Double Integral

$$
V = \int_{0}^{2\pi} \int_{0}^{2} f(r, \theta) \, r\, dr\, d\theta
$$

Step 4: Compute the Inner Integral (with respect to r)

For example, with $f(r,\theta) = 1$:

$$
V = \int_{0}^{2\pi} \left(\int_{0}^{2} r\, dr \right) d\theta
$$

$$
= \int_{0}^{2\pi} \left[\frac{r^2}{2} \right]_0^{2} d\theta = \int_{0}^{2\pi} \frac{4}{2} d\theta
= \int_{0}^{2\pi} 2\, d\theta
$$

$$
= 2 \times 2\pi = 4\pi
$$

which confirms the earlier simple calculation.

Advanced Examples and Variations

Example 1: Volume under $z = y$

Expressed in polar coordinates:

$$
z = y = r \sin \theta
$$

The volume becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} r \sin \theta \, r\, dr\, d\theta = \int_{0}^{2\pi} \sin \theta \left(\int_{0}^{2} r^2 dr \right) d\theta
$$

Compute the inner integral:

$$
\int_{0}^{2} r^2 dr = \left[\frac{r^3}{3} \right]_0^{2} = \frac{8}{3}
$$

Thus,

$$
V = \frac{8}{3} \int_{0}^{2\pi} \sin \theta \, d\theta = \frac{8}{3} \left[-\cos \theta \right]_0^{2\pi} = \frac{8}{3} (-\cos 2\pi + \cos 0) = \frac{8}{3} (-1 + 1) = 0
$$

The volume is zero because the positive and negative contributions cancel out over the symmetric interval. To find the "unsigned" volume, we might take the absolute value or restrict the region.

Example 2: Volume under $z = \sqrt{4 - x^2 - y^2}$ (the upper hemisphere)

Expressed in polar coordinates:

$$
z = \sqrt{4 - r^2}
$$

The volume:

$$
V = \int_{0}^{2\pi} \int_{0}^{2} \sqrt{4 - r^2} \, r\, dr\, d\theta
$$

The inner integral involves:

$$
\int_{0}^{2} r \sqrt{4 - r^2} dr
$$

which can be solved using substitution $u = 4 - r^2$, $du =$

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