Solve The Given Differential Equation. X3y''' 6y = 0 Y(x) = , X > 0

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Introduction to Differential Equations

Differential equations are fundamental in mathematical modeling, describing how quantities change over time or space. They are equations involving derivatives of unknown functions and are crucial in fields such as physics, engineering, biology, and economics. The process of solving differential equations involves finding the function (or functions) that satisfy the given relation.

In this article, we focus on solving a specific third-order linear differential equation:

\[x^3 y''' - 6 y = 0, \quad \text{where } x > 0 \]

This problem presents interesting challenges due to the variable coefficient \(x^3 \) and the order of the differential equation.

Understanding the Given Differential Equation

Equation Overview

The differential equation is:

$$ x^3 y''' - 6 y = 0 $$

- It is a third-order linear differential equation.
- The coefficient x^3 multiplies the highest derivative y''', making it a variable coefficient differential equation.
- The term $-6 y$ involves the function y itself, with no derivatives.

Significance of the Variable Coefficient

Variable coefficient differential equations are more complex than constant coefficient equations. Standard solution methods include:

- Reduction of order
- Power series solutions
- Substitutions to convert into equations with constant coefficients

Given the structure, an effective approach is to consider substitutions that reduce the order or transform the equation into a more manageable form.

Approach to Solving the Differential Equation

Step 1: Recognize the Equation Type

The presence of \(x^3 \) multiplying \(y''' \) suggests considering substitution methods that simplify the variable coefficients. One such approach is to introduce a change of variables, transforming the independent variable or the dependent function.

Step 2: Use of Substitutions

Possible substitutions include:

- \(t = \ln x \) to convert variable coefficients involving powers of \(x \) into constant coefficients.
- Assuming a solution form inspired by the nature of the coefficients, such as power functions or exponential functions.

Let's explore the substitution \(t = \ln x \), which is common when dealing with equations featuring powers of \(x \).

Transformation via Substitution \(t = \ln x \)

Changing Variables

Define:

\[t = \ln x \quad \Rightarrow \quad x = e^t \]

Derivatives with respect to x can be expressed in terms of derivatives with respect to t:

$$
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}
$$

Similarly, higher derivatives involve chain rule applications:

$$
\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2}
$$

$$
\frac{d^3 y}{dx^3} = -\frac{2}{x^3} \frac{dy}{dt} + \frac{3}{x^3} \frac{d^2 y}{dt^2} - \frac{1}{x^3} \frac{d^3 y}{dt^3}
$$

Given the complexity, it is often easier to express derivatives directly:

$$
\frac{dy}{dx} = \frac{1}{x} y'
$$

$$
\frac{d^2 y}{dx^2} = \frac{1}{x^2} (y'' - y')
$$

$$
\frac{d^3 y}{dx^3} = \frac{1}{x^3} (y''' - 3 y'' + 2 y')
$$

where primes denote derivatives with respect to \(t \).

Expressing \(y''' \) in terms of derivatives with respect to \(t \)

From the above:

\[

y''' = x^3 \frac{d^3 y}{dx^3} + 3 x^2 \frac{d^2 y}{dx^2} + 2 x \frac{dy}{dx}

\]

But to avoid confusion, it's more straightforward to directly substitute derivatives into the original equation:

\[

x^3 y''' - 6 y = 0

\]

Express \(y''' \) in terms of derivatives with respect to \(t \). Using the chain rule:

\[

\frac{dy}{dx} = \frac{1}{x} y'

\]

\[

\frac{d^2 y}{dx^2} = \frac{1}{x^2} (y'' - y')

\]

\[

\frac{d^3 y}{dx^3} = \frac{1}{x^3} (y''' - 3 y'' + 2 y')

\]

Therefore:

\[

x^3 y''' = y''' - 3 y'' + 2 y'

\]

Substituting into the original equation:

\[

(y''' - 3 y'' + 2 y') - 6 y = 0

\]

Now, the differential equation in terms of derivatives with respect to \(t \) becomes:

\[

y''' - 3 y'' + 2 y' - 6 y = 0

\]

Simplified Form and Solution Strategy

Reduced Differential Equation

The transformed differential equation is:

\[

y''' - 3 y'' + 2 y' - 6 y = 0

\]

This is a linear, constant-coefficient, third-order ODE.

Characteristic Equation

To solve, assume a solution of the form:

\[

y(t) = e^{r t}

\]

Substitute into the differential equation:

\[

r^3 e^{r t} - 3 r^2 e^{r t} + 2 r e^{r t} - 6 e^{r t} = 0

\]

Divide through by \(e^{r t} \):

\[

r^3 - 3 r^2 + 2 r - 6 = 0

\]

This polynomial is called the characteristic equation:

\[

r^3 - 3 r^2 + 2 r - 6 = 0

\]

Solving the Characteristic Equation

Step 1: Find Roots of the Polynomial

Attempt rational root theorem:

Possible roots are factors of constant term (±1, ±2, ±3, ±6) divided by factors of leading coefficient (1):

\[
\pm 1, \pm 2, \pm 3, \pm 6
\]

Test \(r=1 \):

\[
1 - 3 + 2 - 6 = -6 \neq 0
\]

Test \(r=2 \):

\[
8 - 12 + 4 - 6 = -6 \neq 0
\]

Test \(r=3 \):

\[
27 - 27 + 6 - 6 = 0
\]

Root \(r=3 \) is a root!

Step 2: Polynomial Factorization

Divide the polynomial by \(r-3 \):

Using synthetic division:

Coefficients: 1 | -3 | 2 | -6

Perform synthetic division with root 3:

- Bring down 1

- Multiply by 3: 3

- Add to -3: 0

- Multiply 0 by 3: 0

- Add to 2: 2

- Multiply 2 by 3: 6

- Add to -6: 0

Resulting quadratic:

\[
r^2 + 0 r + 2 = r^2 + 2

\]

Set equal to zero:

\[

r^2 + 2 = 0 \Rightarrow r^2 = -2 \Rightarrow r = \pm i \sqrt{2}

\]

General Solution for \(y(t) \)

The roots are:

\[

r = 3, \quad r = i \sqrt{2}, \quad r = -i \sqrt{2}

\]

The general solution:

\[

y(t) = C_1 e^{3 t} + C_2 e^{i \sqrt{2} t} + C_3 e^{- i \sqrt{2} t}

\]

Using Euler's formula:

\[

e^{i \theta} = \cos \theta + i \sin \theta

\]

The solution simplifies to:

\[

y(t) = C_1 e^{3 t} + D_1 \cos(\sqrt{2} t) + D_2 \sin(\sqrt{2} t)

\]

where \(D_1 \) and \(D_2 \) are constants related to \(C_2 \) and

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $x^3 y''' - 6 y = 0$, where y is a function of x.

What is the domain specified for the solution y(x)?

The domain specified is $x > 0$.

What is the method used to solve the differential equation $x^3 y''' - 6 y = 0$?

Since the equation involves variable coefficients and derivatives, a common method is to try a solution of the form y = x^m and find the corresponding characteristic equation.

How do we assume a solution to solve the differential equation $x^3 y''' - 6 y = 0$?

We assume a power function y = x^m, where m is a constant to be determined.

What is the characteristic equation obtained when substituting y = x^m into the differential equation?

Substituting y = x^m leads to a characteristic equation m(m-1)(m-2) - 6 = 0.

How do we solve the characteristic equation for m?

Solve m(m-1)(m-2) = 6, which simplifies to a cubic equation m^3 - 3m^2 + 2m - 6 = 0, and find its roots.

What are the roots of the characteristic equation, and what do they imply for the solution?

The roots are real and can be found using methods like factoring or numerical approximation; each root corresponds to a term in the general solution y(x).

What is the general solution y(x) for the differential equation?

The general solution is a linear combination of power functions y(x) = C□ x^{m□} + C□ x^{m□} + C□ x^{m□}, where m□, m□, m□ are the roots of the characteristic equation.

How do we apply initial or boundary conditions to find the particular solution?

Using the given y(x) = ... and the condition x > 0, substitute the initial conditions to solve for the constants C□, C□, C□.

What is the significance of the domain x > 0 in solving this differential equation?

The domain x > 0 ensures that the power functions x^m are well-defined and helps avoid issues with negative or zero values of x when applying the solution.

Additional Resources

Solve The Given Differential Equation: An In-Depth Analytical Approach

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Introduction

Differential equations are an essential cornerstone of mathematical modeling, providing the means to describe a vast array of phenomena across physics, engineering, biology, and economics. They translate the language of change into a mathematical form, allowing us to analyze systems' behavior over time or space. Among these, ordinary differential equations (ODEs) – particularly linear ones with constant coefficients – are often the starting point for both theoretical exploration and practical application.

In this article, we delve into a specific third-order differential equation:

$$
x^3 y''' + 6 y = 0, \quad \text{for } x > 0,
$$

where $y = y(x)$ is the unknown function, and the prime notation indicates derivatives with respect to x. Our goal is to analyze, solve, and interpret this equation comprehensively, providing insights into the methods employed and the nature of the solutions.

Understanding the Differential Equation

The Structure and Characteristics

The given differential equation is:

\[

x^3 y''' + 6 y = 0.

\]

Key observations include:

- Order: The highest derivative present is \(y''' \), making this a third-order ODE.
- Variable Coefficient: The coefficient \(x^3 \) is a function of \(x \), implying the equation is not constant coefficient and may require special techniques for solution.
- Homogeneity: The equation is homogeneous, meaning all terms involve \(y \) or its derivatives, and the right side is zero.
- Domain: Defined for \(x > 0 \), which suggests potential singularities at \(x = 0 \), and indicates that solutions should be valid in the positive real domain.

Significance of Variable Coefficients

Variable coefficient differential equations often appear in modeling scenarios where properties change with the independent variable, such as in problems involving spherical coordinates or power-law media. Solving such equations typically involves transformations or special functions.

Strategy for Solution

Recognizing the Type of Equation

Given the structure, the equation resembles a form that could be approached via substitution or reduction of order, especially considering the multiplicative \(x^3 \) term.

Potential Methods

1. Reduction of Order: Useful if initial solutions are known or guessed.

2. Transformations: Recasting the equation into a more manageable form, such as a Cauchy-Euler (equidimensional) equation.
3. Special Function Techniques: Employing known solutions of similar equations, such as Bessel functions or hypergeometric functions, if applicable.

In this case, the key is to recognize whether the equation can be rewritten as a Cauchy-Euler type or if a substitution can convert it into a constant coefficient differential equation.

Transformation to a Standard Form

Dividing Through by x^3

To simplify, divide both sides of the differential equation by x^3, which is valid for $x > 0$:

$$
y''' + \frac{6}{x^3} y = 0.
$$

While this simplifies the structure, the term $\frac{6}{x^3} y$ remains variable in x. However, it suggests exploring a substitution that can eliminate the variable coefficient.

Substitution: $t = \ln x$

A common technique for equations involving powers of x is to use a logarithmic substitution:

$$
t = \ln x \quad \Rightarrow \quad x = e^t,
$$

which transforms derivatives with respect to \(x \) into derivatives with respect to \(t \):

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt},
\]
\[
\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{d^2 y}{dt^2},
\]
\[
\frac{d^3 y}{dx^3} = -\frac{2}{x^3} \frac{dy}{dt} + \frac{3}{x^3} \frac{d^2 y}{dt^2} + \frac{1}{x^3} \frac{d^3 y}{dt^3}.
\]

Substituting into the original equation:

\[
x^3 y''' + 6 y = 0,
\]

becomes:

\[
\left(-2 \frac{dy}{dt} + 3 \frac{d^2 y}{dt^2} + \frac{d^3 y}{dt^3} \right) + 6 y = 0.
\]

Simplify:

\[
\frac{d^3 y}{dt^3} + 3 \frac{d^2 y}{dt^2} - 2 \frac{dy}{dt} + 6 y = 0,
\]

which is a linear constant coefficient third-order differential equation.

Solving the Transformed Equation

Characteristic Equation

The standard approach involves assuming solutions of the form:

$$
y(t) = e^{r t}.
$$

Plugging into the differential equation:

$$
r^3 + 3 r^2 - 2 r + 6 = 0.
$$

This cubic characteristic equation determines the nature of the solutions.

Roots of the Characteristic Equation

To find roots r:

$$
r^3 + 3 r^2 - 2 r + 6 = 0,
$$

we can attempt rational root testing. The Rational Root Theorem suggests possible roots: factors of 6 over factors of 1, i.e., $\pm 1, \pm 2, \pm 3, \pm 6$.

Test $r = 1$:

$$
1 + 3 - 2 + 6 = 8 \neq 0.
$$

Test $r = -1$:

$$
-1 + 3 + 2 + 6 = 10 \neq 0.
$$

Test $r = 2$:

$$
8 + 12 - 4 + 6 = 22 \neq 0.
$$

Test $r = -2$:

$$
-8 + 12 + 4 + 6 = 14 \neq 0.
$$

Test $r = 3$:

$$
27 + 27 - 6 + 6 = 54 \neq 0.
$$

Test \(r = -3 \):

\[
-27 + 27 + 6 + 6 = 12 \neq 0.
\]

Test \(r = 6 \):

\[
216 + 108 - 12 + 6 = 318 \neq 0.
\]

Test \(r = -6 \):

\[
-216 + 108 + 12 + 6 = -90 \neq 0.
\]

No rational roots. Therefore, roots are complex or irrational. Proceed with the cubic formula or numerical approximation.

Roots of the Cubic Equation

Using the cubic formula, the roots are complex conjugates with real parts, or all complex. For brevity, assume roots:

\[
r_1, r_2, r_3,
\]

which may be complex.

General Solution in \(t \)

The general solution:

\[

y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + C_3 e^{r_3 t},

\]

where \(C_1, C_2, C_3 \) are arbitrary constants.

Reverting to \(x \)

Recall:

\[

t = \ln x \quad \Rightarrow \quad e^{r t} = x^{r}.

\]

Thus, the solution in terms of \(x \):

\[

y(x) = C_1 x^{r_1} + C_2 x^{r_2} + C_3 x^{r_3}.

\]

Interpretation of the Solution

Nature of the Roots

Depending on whether roots are real or complex, the solution's form varies:

- Real roots: The solution involves power functions \(x^{r_i} \).
- Complex roots: The solution involves oscillatory behavior with factors like \(x^{\alpha} \) multiplied by sine and cosine functions, stemming from Euler's formula.

General Solution

The comprehensive general solution for \(y(x) \) is:

\[
\boxed{
y(x) = C_1 x^{r_1} + C_2 x^{r_2} + C_3 x^{r_3},
}
\]

where each \(r_i \) is a root of the characteristic polynomial.

Boundary Conditions and Particular Solutions

The initial problem statement specifies:

\[
Y(x) = \quad \text{(presumably an initial condition or boundary value)}.
\]

Without explicit initial conditions, the solution remains in its general form. However, specifying conditions such as \(y(x_0) \), \(y'(x_0) \), \(y''(x_0) \) at some \(x_0 > 0 \) enables solving for the constants \(C_i \).

Special Cases and

Solve The Given Differential Equation X3y 6y 0 Y X X Gt 0

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