- Solve The Initial-value Problem 4" + 4y' + 13y = 0 Y(0) = 3 Y'(0) =3/2

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Introduction to the Differential Equation and Initial Conditions

The differential equation presented, $4y'' + 4y' + 13y = 0$, is a second-order linear homogeneous differential equation with constant coefficients. The initial conditions provided are $Y(0) = 3$ and $Y'(0) = 3/2$, which specify the value of the function and its first derivative at $x = 0$. Solving such problems involves identifying the characteristic equation, solving for its roots, and then formulating the general solution based on the nature of these roots. The initial conditions then allow us to determine the specific constants in the general solution.

This type of problem is fundamental in the study of differential equations, especially because it models many physical systems such as mechanical oscillations, electrical circuits, and other systems where the response depends on initial states.

Step 1: Write the Differential Equation in Standard Form

The differential equation is:

$4y'' + 4y' + 13y = 0$

To simplify, divide through by 4:

$y'' + y' + (13/4)y = 0$

This standard form makes it easier to analyze the characteristic equation.

Step 2: Find the Characteristic Equation

For a second-order linear differential equation with constant coefficients:

$a y'' + b y' + c y = 0$

the characteristic equation is:

$a r^2 + b r + c = 0$

Applying this to our simplified form:

$r^2 + r + (13/4) = 0$

Calculating the Discriminant

The discriminant D of the quadratic is:

$D = b^2 - 4 a c$

$D = 1^2 - 4 \cdot 1 \cdot (13/4) = 1 - 13 = -12$

Since $D < 0$, the roots are complex conjugates, indicating oscillatory behavior.

Step 3: Find the Roots of the Characteristic Equation

Using the quadratic formula:

$r = [-b \pm \sqrt{D}] / (2a)$

Substitute the values:

$r = [-1 \pm \sqrt{(-12)}] / 2$

Recall that $\sqrt{(-12)} = i\sqrt{12} = i\ 2\sqrt{3}$

Therefore:

$r = [-1 \pm i\ 2\sqrt{3}] / 2$

Simplify:

$r = -1/2 \pm i\sqrt{3}$

Interpretation of Roots

The roots are complex conjugates of the form:

$r = \alpha \pm i\beta$

where:

$\alpha = -1/2$

$\beta = \sqrt{3}$

This suggests the general solution will involve exponential decay combined with sinusoidal functions.

Step 4: Write the General Solution

For roots $r = \alpha \pm i\beta$, the general solution of the differential equation is:

$Y(x) = e^{\alpha x} (C_1 \cos(\beta x) + C_2 \sin(\beta x))$

Substituting α and β:

$Y(x) = e^{ - (1/2) x } [C_1 \cos(\sqrt{3} x) + C_2 \sin(\sqrt{3} x)]$

where C_1 and C_2 are constants to be determined.

Step 5: Find the Derivative Y'(x)

Differentiate Y(x):

$Y'(x) = d/dx [e^{ - (1/2) x } (C_1 \cos(\sqrt{3} x) + C_2 \sin(\sqrt{3} x))]$

Apply product rule:

$Y'(x) = e^{ - (1/2) x } d/dx [C_1 \cos(\sqrt{3} x) + C_2 \sin(\sqrt{3} x)] + d/dx [e^{ - (1/2) x }] (C_1 \cos(\sqrt{3} x) + C_2 \sin(\sqrt{3} x))$

Calculate derivatives:

$d/dx [C_1 \cos(\sqrt{3} x)] = - C_1 \sqrt{3} \sin(\sqrt{3} x)$

$d/dx [C_2 \sin(\sqrt{3} x)] = C_2 \sqrt{3} \cos(\sqrt{3} x)$

$d/dx [e^{ - (1/2) x }] = - (1/2) e^{ - (1/2) x }$

Putting it all together:

$Y'(x) = e^{ - (1/2) x } [- C_1 \sqrt{3} \sin(\sqrt{3} x) + C_2 \sqrt{3} \cos(\sqrt{3} x)] - (1/2) e^{ - (1/2) x } [C_1 \cos(\sqrt{3} x) + C_2 \sin(\sqrt{3} x)]$

Factor out $e^{ - (1/2) x }$:

$Y'(x) = e^{ - (1/2) x } [- C_1 \sqrt{3} \sin(\sqrt{3} x) + C_2 \sqrt{3} \cos(\sqrt{3} x) - (1/2) C_1 \cos(\sqrt{3} x) - (1/2) C_2 \sin(\sqrt{3} x)]$

Step 6: Apply Initial Conditions to Find Constants

At $x=0$:

$Y(0) = 3$

$Y'(0) = 3/2$

Calculate Y(0):

$Y(0) = e^{0} [C_1 \cos(0) + C_2 \sin(0)] = 1 [C_1 1 + C_2 0] = C_1$

Therefore:

$C_1 = 3$

Calculate Y'(0):

Y'(0) = e^{0} [- C_1 √3 sin(0) + C_2 √3 cos(0) - (1/2) C_1 cos(0) - (1/2) C_2 sin(0)]

Simplify:

Y'(0) = 1 [0 + C_2 √3 1 - (1/2) 3 1 - 0] = C_2 √3 - (3/2)

Set equal to initial condition:

C_2 √3 - (3/2) = 3/2

Solve for C_2:

C_2 √3 = 3/2 + 3/2 = 3

C_2 = 3 / √3 = √3

Final Solution

Plugging back the constants into the general solution:

Y(x) = e^{ - (1/2) x } [3 cos(√3 x) + √3 sin(√3 x)]

This function satisfies the differential equation and the initial conditions.

Conclusion

The solution of the initial-value problem 4y'' + 4y' + 13y = 0 with Y(0) = 3 and Y'(0) = 3/2 is:

Y(x) = e^{ - (1/2) x } [3 cos(√3 x) + √3 sin(√3 x)]

This solution describes a damped oscillation because of the exponential decay factor combined with sinusoidal components, characteristic of underdamped systems in physics and engineering. The process involved deriving the characteristic roots, formulating the general solution, and applying initial conditions to determine the specific constants, illustrating a standard approach to solving second-order linear differential equations with constant coefficients.

Frequently Asked Questions

What is the initial-value problem given in the differential equation?

The initial-value problem is y'' + 4y' + 13y = 0 with initial conditions Y(0) = 3 and Y'(0) = 3/2.

How do you find the characteristic equation for the differential equation y'' + 4y' + 13y = 0?

The characteristic equation is r^2 + 4r + 13 = 0, obtained by replacing y'' with r^2, y' with r, and y with 1.

What are the roots of the characteristic equation r^2 + 4r + 13 = 0?

The roots are complex: r = -2 ± 3i, found using the quadratic formula.

What is the general solution to the differential equation based on the roots?

The general solution is y(t) = e^{-2t} [C1 cos(3t) + C2 sin(3t)].

How do you determine the constants C1 and C2 using initial conditions?

Substitute t = 0 into y(t) and y'(t), then solve the resulting system of equations using y(0) = 3 and y'(0) = 3/2.

What is the explicit solution to the initial-value problem after applying initial conditions?

The solution is y(t) = e^{-2t} [3 cos(3t) + 0.5 sin(3t)].

Why is the solution involving exponential decay and oscillation?

Because the roots are complex conjugates with negative real parts, leading to oscillatory behavior multiplied by exponential decay.

Additional Resources

Solve The Initial-Value Problem 4y'' + 4y' + 13y = 0, with Y(0) = 3 and Y'(0) = 3/2

Introduction

The realm of differential equations is foundational to understanding the behavior of various physical, biological, and engineering systems. Among these, second-order linear differential equations with constant coefficients are particularly prevalent, modeling oscillatory, damping, and other dynamic phenomena. The initial-value problem (IVP) we examine here—4y'' + 4y' + 13y = 0, with initial conditions Y(0) = 3 and Y'(0) = 3/2—serves as an illustrative case of solving homogeneous linear differential equations with constant coefficients, highlighting key analytical techniques and interpretive insights.

This comprehensive review aims to explore the methodologies involved, analyze the solution's structure, and interpret its implications within a broader mathematical context.

Formulation of the Problem

The differential equation under scrutiny is:

$$
4 y'' + 4 y' + 13 y = 0
$$

subject to initial conditions:

$$
Y(0) = 3, \quad Y'(0) = \frac{3}{2}
$$

The primary goal is to find a function $Y(t)$ satisfying both the differential equation and the initial conditions, and to understand the behavior of this solution over time.

Step 1: Standard Form and Characteristic Equation

Recasting the Differential Equation

To facilitate solution, divide through by 4 to normalize the leading coefficient:

$$
y'' + y' + \frac{13}{4} y = 0
$$

The normalized form:

$$
\boxed{
y'' + y' + \frac{13}{4} y = 0
}
$$

This is a homogeneous linear differential equation with constant coefficients, conducive to solving via characteristic equations.

Deriving the Characteristic Equation

Assuming solutions of the form $y(t) = e^{r t}$, substitute into the differential equation:

$$
r^2 e^{r t} + r e^{r t} + \frac{13}{4} e^{r t} = 0
$$

Dividing through by $e^{r t} \neq 0$:

$$
r^2 + r + \frac{13}{4} = 0
$$

The characteristic (auxiliary) equation:

$$
r^2 + r + \frac{13}{4} = 0
$$

Step 2: Solving the Characteristic Equation

The roots of the quadratic are:

$$
r = \frac{-b \pm \sqrt{b^2 - 4 a c}}{2 a}
$$

where $a=1$, $b=1$, $c=\frac{13}{4}$.

Calculate discriminant:

$$
\Delta = b^2 - 4ac = 1^2 - 4 \times 1 \times \frac{13}{4} = 1 - 13 = -12
$$

Since the discriminant is negative, roots are complex conjugates:

$$
r = \frac{-1 \pm \sqrt{-12}}{2} = \frac{-1 \pm i \sqrt{12}}{2}
$$

Simplify $\sqrt{12} = 2 \sqrt{3}$:

$$
r = \frac{-1 \pm i 2 \sqrt{3}}{2} = -\frac{1}{2} \pm i \sqrt{3}
$$

Step 3: General Solution

The general solution for the differential equation with complex conjugate roots is:

$$
Y(t) = e^{\alpha t} \left[C_1 \cos(\beta t) + C_2 \sin(\beta t) \right]
$$

where:

$$
\alpha = -\frac{1}{2}, \quad \beta = \sqrt{3}
$$

Thus:

$$
\boxed{
Y(t) = e^{-\frac{1}{2} t} \left[C_1 \cos(\sqrt{3} t) + C_2 \sin(\sqrt{3} t) \right]
}
$$

Step 4: Applying Initial Conditions

To find C_1 and C_2, compute $Y(0)$ and $Y'(0)$.

At $t=0$:

$$
Y(0) = e^0 [C_1 \cos 0 + C_2 \sin 0] = C_1 \times 1 + C_2 \times 0 = C_1
$$

Given $Y(0) = 3$, so:

$$
C_1 = 3
$$

Derivative $Y'(t)$:

$$
Y'(t) = \frac{d}{dt} \left[e^{-\frac{1}{2} t} (C_1 \cos(\sqrt{3} t) + C_2 \sin(\sqrt{3} t)) \right]
$$

Apply product rule:

$$
Y'(t) = e^{-\frac{1}{2} t} \left[-\frac{1}{2} (C_1 \cos(\sqrt{3} t) + C_2 \sin(\sqrt{3} t)) + (- C_1 \sqrt{3} \sin(\sqrt{3} t) + C_2 \sqrt{3} \cos(\sqrt{3} t)) \right]
$$

Simplify:

$$
Y'(t) = e^{-\frac{1}{2} t} \left[-\frac{1}{2} C_1 \cos(\sqrt{3} t) - \frac{1}{2} C_2 \sin(\sqrt{3} t) - C_1 \sqrt{3} \sin(\sqrt{3} t) + C_2 \sqrt{3} \cos(\sqrt{3} t) \right]
$$

At $t=0$:

$$
Y'(0) = e^{0} \left[-\frac{1}{2} C_1 \times 1 - \frac{1}{2} C_2 \times 0 - C_1 \sqrt{3} \times 0 + C_2 \sqrt{3} \times 1 \right]
$$

Simplify:

$$
Y'(0) = -\frac{1}{2} C_1 + C_2 \sqrt{3}

Given $Y'(0) = \frac{3}{2}$, and $C_1 = 3$:

$$
\frac{3}{2} = -\frac{1}{2} \times 3 + C_2 \sqrt{3}
$$

Calculate:

$$
\frac{3}{2} = -\frac{3}{2} + C_2 \sqrt{3}
$$

Add $\frac{3}{2}$ to both sides:

$$
\frac{3}{2} + \frac{3}{2} = C_2 \sqrt{3}
$$

$$
3 = C_2 \sqrt{3}
$$

Solve for C_2:

$$
C_2 = \frac{3}{\sqrt{3}} = \sqrt{3}
$$

Final Solution

$$
\boxed{
Y(t) = e^{-\frac{1}{2} t} \left[3 \cos(\sqrt{3} t) + \sqrt{3} \sin(\sqrt{3} t) \right]
}
$$

This solution satisfies both the differential equation and the initial conditions.

Analysis and Interpretation

Behavior Over Time

The exponential decay factor $e^{-\frac{1}{2} t}$ indicates damping, leading the oscillations to diminish as $t \to \infty$. The sinusoidal components reflect oscillatory behavior with angular frequency $\sqrt{3}$, characteristic of underdamped systems.

Oscillatory Dynamics

The combination of sine and cosine functions signifies phase shifts and amplitude modulations dictated by initial conditions. The amplitude envelope decays exponentially, consistent with physical systems exhibiting damping.

Special Cases

- Zero initial velocity would imply solving for different constants.
- Purely oscillatory solutions (no damping) occur if the real part of roots is zero; here, damping is evident.

Broader Context and Applications

Second-order linear differential equations of this form are central to modeling harmonic oscillators, electrical RLC circuits, mechanical vibrations, and even quantum systems. The damping term captured by the exponential decay influences the stability and long-term behavior of these systems.

Understanding the solution structure aids in designing systems with desired oscillatory or damping properties, optimizing initial conditions, and predicting future states.

Summary

The initial-value problem $4y'' + 4y' + 13y = 0$ with initial conditions $Y(0) = 3$ and $Y'(0) = 3/2$ has the explicit solution:

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\[
\boxed{
Y(t) = e^{-\frac{1}{2} t} \left[ 3 \cos
```

Solve The Initial Value Problem 4 4y 13y 0 Y 0 3 Y 0 3 2

Solve The Initial Value Problem 4 4y 13y 0 Y 0 3 Y 0 3 2

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