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Introduction

Solving initial value problems (IVPs) involving differential equations is a fundamental aspect of applied mathematics, physics, and engineering. The problem at hand, ty'' - 2y' + Ty = T with initial conditions Y(0) = 1 and Y'(0) = 0, presents a second-order linear differential equation with variable coefficients. Addressing such equations requires a systematic approach, often involving methods such as substitution, variation of parameters, or power series, depending on the form of the coefficients.

In this comprehensive article, we will explore the step-by-step process to solve this initial value problem, examine the underlying theory, and interpret the solution in context. This detailed guide aims to enhance understanding for students, educators, and professionals interested in differential equations.

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1. Understanding the Differential Equation

The given differential equation is:

$$ t y'' - 2 y' + T y = T $$

with initial conditions:

$$ Y(0) = 1 $$
$$ Y'(0) = 0 $$

Note: In the original problem statement, the variables are expressed as y and Y; typically, lowercase y denotes the unknown function, but for clarity, we will maintain consistent notation.

Important highlights:

- The coefficient of y'' is t, making this a variable coefficient differential equation.
- The right-hand side is T, which appears to be a variable or parameter. For this problem, assuming T is a constant parameter (or possibly a variable), but since the initial conditions are specified at $t=0$, and the differential equation involves t, it's most plausible that T is a constant.

For this article, we interpret T as a constant parameter, not the variable t. If T were variable, the problem would be different.

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2. Restating the Problem

The differential equation simplifies to:

$$ t y'' - 2 y' + T y = T $$

with initial conditions:

$$ y(0) = 1 $$
$$ y'(0) = 0 $$

Our goal is to find a function $y(t)$ satisfying this equation and initial conditions.

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3. Approach to the Solution

Given the structure of the differential equation, the following strategies are suitable:

- Identify the type of differential equation: The presence of $t y''$ suggests an equation with variable coefficients, possibly similar to an Euler-Cauchy equation.
- Attempt to find the homogeneous solution: Solve the associated homogeneous equation.
- Find a particular solution: Use methods like variation of parameters, undetermined coefficients, or power series.
- Apply initial conditions: Determine the constants in the general solution.

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4. Solving the Homogeneous Equation

The homogeneous equation associated with the given differential equation is:

$$ t y'' - 2 y' + T y = 0 $$

This is a form of Euler-Cauchy (or equidimensional) equation when coefficients are powers of t. Let's analyze it:

$$ t y'' - 2 y' + T y = 0 $$

Dividing through by t (assuming $t \neq 0$):

$$ y'' - \frac{2}{t} y' + \frac{T}{t} y = 0 $$

This form suggests solutions involving powers of t. To proceed, consider the substitution:

$$y(t) = t^{m}$$

which is typical for Euler-type equations. Let's verify whether this substitution helps.

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5. Homogeneous Solution via Power Substitution

Assuming:

$$y(t) = t^{m}$$

then:

$$y' = m t^{m - 1}$$
$$y'' = m (m - 1) t^{m - 2}$$

Substitute into the homogeneous equation:

$$t \cdot m (m - 1) t^{m - 2} - 2 \cdot m t^{m - 1} + T t^{m} = 0$$

Simplify:

$$m (m - 1) t^{m - 1} - 2 m t^{m - 1} + T t^{m} = 0$$

Divide through by $t^{m - 1}$:

$$m (m - 1) - 2 m + T t = 0$$

This simplifies to:

$$m (m - 1) - 2 m + T t = 0$$

or:

$$(m^{2} - m) - 2 m + T t = 0$$

$$m^{2} - m - 2 m + T t = 0$$

$$m^{2} - 3 m + T t = 0$$

This indicates that unless $T t = 0$, the assumption $y(t) = t^{m}$ does not yield a solution in terms of algebraic expressions for arbitrary t. Therefore, the Euler-Cauchy method leads to complications because of the nonhomogeneous term involving $T t$.

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6. Transforming the Equation

Given the limitations above, an effective approach involves transforming the differential equation into a more manageable form. Let's consider the substitution:

$$ y(t) = u(t) $$

and look for an integrating factor or substitution to reduce the order.

Alternatively, since the equation involves $t y''$, it resembles a form of equation with variable coefficients that can sometimes be tackled via substitution $z = y'$.

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7. Using Variation of Parameters

Since the homogeneous solution seems complex to find directly, an alternative is to:

- Find the homogeneous solution using known methods.
- Find a particular solution using variation of parameters.

Let's attempt to find the homogeneous solution again with an alternative approach:

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8. Homogeneous Equation as a Standard Form

Rewrite the homogeneous equation:

$$ t y'' - 2 y' + T y = 0 $$

Divide through by t:

$$ y'' - \frac{2}{t} y' + \frac{T}{t} y = 0 $$

This is a second-order linear differential equation with variable coefficients similar to the Bessel or confluent hypergeometric equations.

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9. Recognizing the Equation Type

The form:

$$ y'' + p(t) y' + q(t) y = 0 $$

where:

$$ p(t) = - \frac{2}{t} $$
$$ q(t) = \frac{T}{t} $$

is characteristic of equations whose solutions involve special functions, such as Bessel functions.

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10. Transforming to Standard Bessel Equation

Let's attempt a substitution to match the standard form of Bessel's differential equation.

Suppose:

$$ y(t) = t^{k} v(t) $$

Choosing k to simplify the coefficients.

Calculate derivatives:

$$ y' = k t^{k - 1} v(t) + t^{k} v'(t) $$
$$ y'' = k (k - 1) t^{k - 2} v(t) + 2 k t^{k - 1} v'(t) + t^{k} v''(t) $$

Substitute into the original homogeneous equation:

$$ t \left[k (k - 1) t^{k - 2} v + 2 k t^{k - 1} v' + t^{k} v'' \right] - 2 \left[k t^{k - 1} v + t^{k} v' \right] + T t^{k} v = 0 $$

Simplify each term:

- First term:

$$ t \cdot k(k - 1) t^{k - 2} v = k (k - 1) t^{k - 1} v $$

- Second term:

$$ t \cdot 2 k t^{k - 1} v' = 2 k t^{k} v' $$

- Third term:

$$ t \cdot t^{k} v'' = t^{k + 1} v'' $$

- Fourth term:

$$ -2 k t^{k - 1} v $$

- Fifth term:

$$ -2 t^{k} v' $$

- Sixth term:

$$ T t^{k} v $$

Now, combine:

$$ k(k - 1) t^{k - 1} v + 2 k t^{k} v' + t^{k + 1} v'' - 2 k t^{k - 1} v - 2 t^{k} v' + T t^{k} v = 0 $$

Group similar terms:

$$ \left[k(k - 1) t^{k - 1} v - 2 k t^{k - 1} v \right] + \left[2 k t^{k} v' - 2 t^{k} v' \right] + t^{k + 1} v'' + T t^{k} v = 0 $$

Simplify:

\[[k(k - 1) - 2k]

Frequently Asked Questions

What is the initial value problem given in the differential equation?

The initial value problem is ty'' - 2y' + ty = T with initial conditions Y(0) = 1 and Y'(0) = 0.

How do you approach solving the differential equation ty'' - 2y' + ty = T?

You can approach it by recognizing it as a linear nonhomogeneous differential equation, possibly using an integrating factor, variation of parameters, or transforming it into a standard form for solution.

What is the significance of the initial conditions Y(0) = 1 and Y'(0) = 0 in solving this problem?

They are used to determine the constants of integration after solving the general solution of the differential equation, ensuring the particular solution fits the initial conditions.

Can the differential equation ty'' - 2y' + ty = T be simplified or transformed to a more familiar form?

Yes, by dividing through by t (for t ≠ 0), it can be rewritten as y'' - (2/t)y' + y = 1, which resembles a form suitable for methods like variation of parameters.

What method is most suitable for solving the nonhomogeneous differential equation ty'' - 2y' + ty = T?

Variation of parameters is suitable here, especially since the nonhomogeneous term T can be viewed as a forcing function, and the equation resembles a Cauchy-Euler type after transformation.

What is the general solution to the homogeneous part of the differential equation?

The homogeneous equation ty'' - 2y' + ty = 0 can be transformed into a standard form and solved, often leading to solutions involving special functions or power series, depending on the method used.

How do you find the particular solution for the nonhomogeneous differential equation?

Using variation of parameters or the method of undetermined coefficients, you can find a particular solution that accounts for the nonhomogeneous term T.

Once the general solution is found, how are the initial conditions applied to determine the specific solution?

You substitute t=0 into the general solution and its derivative, then solve the resulting equations using Y(0) = 1 and Y'(0) = 0 to find the constants of integration.

Additional Resources

Solve The Initial Value Problem: ty'' - 2y' + Ty = T, Y(0) = 1, Y'(0)=0

When encountering differential equations, especially initial value problems (IVPs), the goal is to find a function y(t) that satisfies both the differential equation and the given initial conditions. The problem at hand—ty'' - 2y' + Ty = T with initial conditions Y(0) = 1 and Y'(0)=0—is a second-order linear nonhomogeneous differential equation with variable coefficients. Solving such an equation involves a combination of analytical techniques, including substitution, the method of solving homogeneous equations, and particular solutions. This review provides a comprehensive exploration of solving this IVP, highlighting methods, step-by-step procedures, and insights into the problem's structure.

Understanding the Differential Equation

The given differential equation:

$$ ty'' - 2y' + Ty = T $$

poses an interesting challenge because of its variable coefficient t multiplying the second derivative y''. Such equations often fall into the category of Cauchy-Euler (or equidimensional) equations, especially if the variable coefficient is a power of t. Here, however, the coefficient is simply t, which suggests that substitution or transformation techniques may be employed to convert the equation into a more manageable form.

Key features:
- Variable coefficient t multiplying the highest derivative.
- Nonhomogeneous term T, which is likely a function of t (assuming T is a parameter or function of t).
- Initial conditions are provided at $t=0$, which requires careful handling due to potential singularities at $t=0$.

Note: The notation T in the original problem may indicate a parameter or function of t. For clarity, here, we will assume T is a constant parameter, but the solution process can be adapted if T is a function.

Reformulating the Problem

To analyze and solve this IVP, it helps to understand the structure of the differential equation better. If T is considered a constant parameter, then the equation becomes:

$$ t y'' - 2 y' + T y = T $$

This resembles an Euler-Cauchy type differential equation but with non-constant coefficients because of the variable t. The standard Euler-Cauchy form is:

$$ t^n y^{(n)} + \dots + a_1 t y' + a_0 y = 0 $$

In our case, the presence of $t y''$ aligns with Euler's form, but the additional nonhomogeneous term T and the coefficients require particular attention.

Homogeneous Equation Analysis

Before tackling the nonhomogeneous part, solving the homogeneous version:

$$ t y'' - 2 y' + T y = 0 $$

sets the foundation for the general solution. This involves assuming a solution of the form:

$$ y_h = t^m $$

which is standard for Euler equations. Substituting into the homogeneous equation:

$$
t \cdot \frac{d^2}{dt^2}(t^m) - 2 \frac{d}{dt}(t^m) + T t^m = 0
$$

Calculating derivatives:

$$
\frac{d}{dt}(t^m) = m t^{m-1}
$$
$$
\frac{d^2}{dt^2}(t^m) = m (m-1) t^{m-2}
$$

Substituting back:

$$
t \cdot m (m-1) t^{m-2} - 2 m t^{m-1} + T t^m = 0
$$

Simplify each term:

$$
m (m-1) t^{m-1} - 2 m t^{m-1} + T t^{m} = 0
$$

Factor t^{m-1}:

$$
t^{m-1} [m (m-1) - 2 m + T t] = 0
$$

Dividing through by t^{m-1}:

$$
m (m-1) - 2 m + T t = 0
$$

This simplifies to:

$$
m^2 - m - 2 m + T t = 0
$$
$$
m^2 - 3 m + T t = 0
$$

However, since t appears explicitly in the last term, the initial assumption $y = t^m$ is insufficient unless T is zero or constant, and the equation reduces to a form where the coefficients are independent of t.

Alternative approach: Recognize that the equation resembles an Euler-Cauchy form if rewritten, or else consider substitution methods.

Transforming the Equation using Substitution

Given the complications, a more effective approach is to consider a substitution to eliminate the variable coefficient:

Method 1: Change of variable $t = e^x$

- Define $t = e^x$, which implies $x = \ln t$.
- Express derivatives with respect to x:

$$
\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{1}{t} \frac{dy}{dx}
$$

$$
\frac{d^2 y}{dt^2} = \frac{d}{dt} \left(\frac{1}{t} \frac{dy}{dx} \right) = -\frac{1}{t^2} \frac{dy}{dx} + \frac{1}{t} \frac{d}{dt} \left(\frac{dy}{dx} \right)
$$

But this approach complicates the derivatives further. Alternatively, a more straightforward substitution is to consider the substitution:

Method 2: Let $z(t) = y(t)$ and attempt to reduce the order or find an integrating factor.

Applying the Method of Reduction of Order or Variation of Parameters

Given the complexities, the most effective techniques are:

- Method of Frobenius or power series expansion near $t=0$, especially since initial conditions are at $t=0$.
- Variation of parameters to find a particular solution once the homogeneous solution is known.

Step 1: Find the homogeneous solutions

Assuming the equation reduces to a form similar to Euler's, and considering the coefficients, the homogeneous equation:

$$ t y'' - 2 y' + T y = 0 $$

has solutions of the form:

$$
y_h(t) = t^\lambda
$$

Substituting into the homogeneous equation:

$$
t \cdot \lambda (\lambda - 1) t^{\lambda - 2} - 2 \lambda t^{\lambda - 1} + T t^{\lambda} = 0
$$

Simplify:

$$
\lambda (\lambda - 1) t^{\lambda - 1} - 2 \lambda t^{\lambda - 1} + T t^{\lambda} = 0
$$

Divide through by $t^{\lambda - 1}$:

$$
\lambda (\lambda - 1) - 2 \lambda + T t = 0
$$

Again, the explicit dependence on t suggests that the solution is not purely a power function unless $T=0$. Therefore, for the general case, this indicates that the solution space may involve special functions or requires a different approach.

Alternative Solution Approach: Power Series Method

Because the equation has variable coefficients and initial conditions at $t=0$, the power series method is suitable.

Assumption:

$$
Y(t) = \sum_{n=0}^{\infty} a_n t^{n + r}
$$

where r is the indicial exponent determined by the behavior near $t=0$.

Procedure:

1. Compute derivatives Y' and Y''.
2. Substitute into the differential equation.
3. Collect like powers of t.
4. Find the recurrence relation for the coefficients a_n.
5. Apply initial conditions to determine a_0, a_1, etc.

This approach is rigorous but lengthy and provides insights into the solution's structure near $t=0$.

Handling Initial Conditions

Given:

$$
Y(0) = 1, \quad Y'(0) = 0
$$

Applying these to the power series expansion:

$$
Y(t) = a_0 + a_1 t + a_2 t^2 + \dots
$$

then:

$$
Y(0) = a_0 = 1
$$

and

$$
Y'(t) = a_1 + 2 a_2 t + \dots
$$

so

$$
Y'(0) = a_1 = 0
$$

These initial condition values help determine the constants in the series solution.

Particular Solution and Complete Solution