Solve The Initial-value Problem: Y" - 4y' + 8y = 0, Y(0) = 1, Y'(0) = 2.

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Understanding how to solve differential equations with initial conditions is a fundamental aspect of applied mathematics and engineering. In this article, we focus on solving the initial-value problem (IVP):

$Y'' - 4Y' + 8Y = 0$, with initial conditions $Y(0) = 1$ and $Y'(0) = 2$.

This type of differential equation is a second-order linear homogeneous differential equation with constant coefficients. Solving it involves several steps, including finding the characteristic equation, determining the general solution, and applying the initial conditions to find specific constants.

This comprehensive guide will walk you through each step in detail, providing insights into the theory and practical methods used. Whether you are a student or a professional, understanding this process enhances your ability to tackle similar differential equations.

Understanding the Differential Equation

The differential equation under consideration is:

$Y'' - 4Y' + 8Y = 0$

- Order: Second-order because the highest derivative is Y''.
- Linearity: Linear due to the sum of derivatives and the function itself, with constant coefficients.
- Homogeneity: No term involving Y explicitly multiplied by a function of x; the right-hand side is zero.

This type of differential equation often appears in physics, engineering, and applied sciences, modeling phenomena like oscillations, electrical circuits, and mechanical vibrations.

Step 1: Write the Characteristic Equation

The method for solving linear homogeneous equations with constant coefficients involves assuming a solution of the form:

$Y(x) = e^{rx}$

where r is a constant to be determined.

Plugging $Y = e^{rx}$ into the differential equation gives:

$r^2 e^{rx} - 4r e^{rx} + 8 e^{rx} = 0$

Dividing through by e^{rx} (which is never zero), we obtain the characteristic equation:

$r^2 - 4r + 8 = 0$

This quadratic equation determines the roots that will define the general solution.

Step 2: Find the Roots of the Characteristic Equation

Solve the quadratic:

$r^2 - 4r + 8 = 0$

Using the quadratic formula:

$r = [4 \pm \sqrt{(16 - 418)}] / 2$

Calculate the discriminant:

$\Delta = 16 - 32 = -16$

Since the discriminant is negative, the roots are complex conjugates:

$r = [4 \pm \sqrt{(-16)}] / 2$

$r = [4 \pm 4i] / 2$

Simplify:

$r = 2 \pm 2i$

These roots are complex conjugates with real part 2 and imaginary part 2.

Step 3: Write the General Solution

For complex roots $r = \alpha \pm \beta i$, the general solution of the differential equation is:

$Y(x) = e^{\alpha x} [C_1 \cos(\beta x) + C_2 \sin(\beta x)]$

In our case:

$\alpha = 2$

$\beta = 2$

Therefore, the general solution is:

$Y(x) = e^{2x} [C_1 \cos(2x) + C_2 \sin(2x)]$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Step 4: Apply Initial Conditions to Find Constants

Given initial conditions:

$Y(0) = 1$

$Y'(0) = 2$

Let's compute $Y(0)$:

$Y(0) = e^{0} [C_1 \cos(0) + C_2 \sin(0)] = 1 [C_1 1 + C_2 0] = C_1$

So:

$C_1 = 1$

Next, compute $Y'(x)$:

$Y'(x) =$ derivative of $Y(x)$

Using the product rule:

$Y'(x) =$ derivative of $e^{2x} [C_1 \cos(2x) + C_2 \sin(2x)]$

$Y'(x) = e^{2x}\, 2\, [C_1 \cos(2x) + C_2 \sin(2x)] + e^{2x} [-2 C_1 \sin(2x) + 2 C_2 \cos(2x)]$

Simplify:

$Y'(x) = e^{2x} [2 C_1 \cos(2x) + 2 C_2 \sin(2x) - 2 C_1 \sin(2x) + 2 C_2 \cos(2x)]$

Combine like terms:

$Y'(x) = e^{2x} [(2 C_1 \cos(2x) + 2 C_2 \cos(2x)) + (2 C_2 \sin(2x) - 2 C_1 \sin(2x))]$

$Y'(x) = e^{2x} [2 (C_1 + C_2) \cos(2x) + 2 (C_2 - C_1) \sin(2x)]$

Evaluate at $x=0$:

$Y'(0) = e^{0} [2 (C_1 + C_2) \cos(0) + 2 (C_2 - C_1) \sin(0)] = 1 [2 (C_1 + C_2)\, 1 + 2 (C_2 - C_1)\, 0] = 2 (C_1 + C_2)$

Given $Y'(0) = 2$, and $C_1 = 1$:

$2 (1 + C_2) = 2$

Divide both sides by 2:

$1 + C_2 = 1$

Solve for C_2:

$C_2 = 0$

Final Solution

Substitute the constants back into the general solution:

$Y(x) = e^{2x} [C_1 \cos(2x) + C_2 \sin(2x)] = e^{2x} [1 \cos(2x) + 0 \sin(2x)] = e^{2x} \cos(2x)$

Thus, the unique solution to the initial-value problem is:

$Y(x) = e^{2x} \cos(2x)$

Additional Insights and Applications

Understanding this solution provides valuable insights into the behavior of systems modeled by such differential equations:

- Oscillatory Growth: The exponential term e^{2x} indicates an exponential growth, while cosine introduces oscillations.
- Physical Systems: This solution can model phenomena like damped or amplified oscillations in mechanical or electrical systems.
- Stability: The positive real part in the exponential indicates the solution grows without bound as x increases, implying instability in some contexts.

Summary of Key Steps

To summarize, solving the initial-value problem involved:

1. Formulating the characteristic equation: $r^2 - 4r + 8 = 0$
2. Finding roots: Complex conjugates $r = 2 \pm 2i$
3. Writing the general solution: $Y(x) = e^{2x} [C_1 \cos(2x) + C_2 \sin(2x)]$
4. Applying initial conditions: $Y(0) = 1$ and $Y'(0) = 2$, to find $C_1 = 1$, $C_2 = 0$
5. Final solution: $Y(x) = e^{2x} \cos(2x)$

Conclusion

Solving second-order linear homogeneous differential equations with constant coefficients and initial conditions is a systematic process rooted in characteristic equations and exponential-trigonometric solutions. This example demonstrates how to approach such problems methodically, interpret the roots of the characteristic equation, and use initial conditions to determine specific solutions.

Mastering these techniques is essential for students and professionals working in engineering, physics, and applied mathematics, where such equations frequently model real-world systems. Practice with different types of equations and initial conditions will deepen your understanding and proficiency in differential equations.

FAQs

Q1: What happens if the roots of the characteristic equation are real and repeated?
Answer: The general solution involves terms like e^{rx} and $x e^{rx}$ to account for multiplicity.

Q2: How can I verify that my solution satisfies the initial conditions?
Answer: Substitute $x=0$ into your solution and its derivative; verify that they match the initial values.

Q3: Can this method be applied to non-homogeneous equations?
Answer: Yes, but it involves finding a particular solution in addition to the homogeneous solution.

Q4: Are there software tools to help solve such differential equations?
Answer: Yes, tools like WolframAlpha, MATLAB, Mathematica, and online calculators can assist in solving and verifying solutions.

Keywords: differential equations, initial-value problem, second-order linear differential equation, characteristic equation, complex roots, exponential oscillations, mathematical modeling, solution methods

Frequently Asked Questions

What is the differential equation given in the initial-value problem?

The differential equation is $Y'' - 4Y' + 8Y = 0$.

What are the initial conditions provided for the problem?

The initial conditions are $Y(0) = 1$ and $Y'(0) = 2$.

How do you find the characteristic equation of the differential equation?

The characteristic equation is r^2 - 4r + 8 = 0.

What are the roots of the characteristic equation r^2 - 4r + 8 = 0?

The roots are complex: $r = 2 \pm i\sqrt{2}$.

What is the general solution to the differential equation?

The general solution is Y(t) = e^{2t} [C_1 cos(√2 t) + C_2 sin(√2 t)].

How are the constants C_1 and C_2 determined using the initial conditions?

By substituting t=0 and the initial conditions into the general solution and its derivative to solve for C_1 and C_2.

What is the value of C_1 based on the initial condition $Y(0) = 1$?

$C_1 = 1$, since at t=0, Y(0) = e^{0} [C_1 1 + C_2 0] = C_1 = 1.

How do you find C_2 using the initial derivative condition $Y'(0) = 2$?

Differentiate the general solution to find Y'(t), then substitute t=0 and Y'(0)=2 to solve for C_2.

What is the explicit solution to the initial-value problem?

The solution is Y(t) = e^{2t} [cos(√2 t) + (1/√2) sin(√2 t)].

Additional Resources

Solve The Initial-value Problem: Y" - 4y' + 8y = 0, Y(0) = 1, Y'(0) = 2 is a classic example of solving a second-order linear homogeneous differential equation with constant coefficients. This type of problem is fundamental in understanding the behavior of systems modeled by differential equations, such as mechanical vibrations, electrical circuits, and many physical phenomena. In this comprehensive review, we will explore the steps involved in solving this initial-value problem (IVP), analyze the solution's characteristics, and discuss the methods and techniques that can be applied to similar problems.

Understanding the Differential Equation: Y" - 4y' + 8y = 0

The differential equation is a second-order linear homogeneous differential equation with constant coefficients. It can be expressed as:

$Y'' - 4Y' + 8Y = 0$

This form indicates that the solution will involve characteristic equations, which provide a systematic method for finding the general solution.

Characteristic Equation

The first step in solving this type of differential equation is to assume a solution of the form:

$Y(t) = e^{rt}$

where r is a constant to be determined.

Substituting into the differential equation yields:

$r^2 e^{rt} - 4r e^{rt} + 8 e^{rt} = 0$

Dividing through by e^{rt} (which is never zero):

$r^2 - 4r + 8 = 0$

This quadratic equation is known as the characteristic equation of the differential equation.

Solving the Characteristic Equation

The quadratic:

$r^2 - 4r + 8 = 0$

can be solved using the quadratic formula:

$r = [4 \pm \sqrt{(16 - 32)}] / 2$

Simplifying:

$r = [4 \pm \sqrt{(-16)}] / 2$

$r = [4 \pm 4i] / 2$

$r = 2 \pm 2i$

The roots are complex conjugates, indicating that the general solution will involve exponential functions combined with sinusoidal functions.

General Solution of the Differential Equation

The nature of the roots (complex conjugates) leads to the following general solution:

$Y(t) = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t)$

where α = real part of the roots, β = imaginary part of the roots, and C_1, C_2 are arbitrary constants.

For our roots:

$\alpha = 2, \beta = 2$

Therefore, the general solution becomes:

$Y(t) = e^{2t} (C_1 \cos 2t + C_2 \sin 2t)$

Applying Initial Conditions

Given initial conditions:

$Y(0) = 1$

$Y'(0) = 2$

We will use these to solve for C_1 and C_2.

First, evaluate $Y(0)$:

$$Y(0) = e^{0} (C_1 \cos 0 + C_2 \sin 0) = C_1 1 + C_2 0 = C_1$$

Since $Y(0) = 1$:

$$C_1 = 1$$

Next, find $Y'(t)$:

$$Y(t) = e^{2t} (C_1 \cos 2t + C_2 \sin 2t)$$

Using product rule:

$$Y'(t) = d/dt [e^{2t} (C_1 \cos 2t + C_2 \sin 2t)]$$

$$Y'(t) = e^{2t} 2 (C_1 \cos 2t + C_2 \sin 2t) + e^{2t} (-2 C_1 \sin 2t + 2 C_2 \cos 2t)$$

Simplify:

$$Y'(t) = e^{2t} [2 C_1 \cos 2t + 2 C_2 \sin 2t - 2 C_1 \sin 2t + 2 C_2 \cos 2t]$$

Group terms:

$$Y'(t) = e^{2t} [(2 C_1 \cos 2t + 2 C_2 \cos 2t) + (-2 C_1 \sin 2t + 2 C_2 \sin 2t)]$$

$$= e^{2t} [2 (C_1 + C_2) \cos 2t + 2 (-C_1 + C_2) \sin 2t]$$

Now, evaluate at $t=0$:

$$Y'(0) = e^{0} [2 (C_1 + C_2) \cos 0 + 2 (-C_1 + C_2) \sin 0] = 1 [2 (C_1 + C_2) 1 + 0] = 2 (C_1 + C_2)$$

Given that $Y'(0) = 2$:

$$2 (1 + C_2) = 2$$

Dividing both sides by 2:

$$1 + C_2 = 1$$

Thus:

$$C_2 = 0$$

Final Solution:

$Y(t) = e^{2t} (\cos 2t)$

Features and Analysis of the Solution

The explicit solution $Y(t) = e^{2t} \cos 2t$ provides insight into the behavior of the system modeled by this differential equation.

Features of the Solution

- Exponential Growth: The factor e^{2t} indicates that the solution exhibits exponential growth as t increases, reflecting an unstable or amplifying system.

- Oscillatory Behavior: The cosine term $\cos 2t$ introduces oscillations with a frequency of 2 radians per unit time, representing periodic or wave-like phenomena.

- Combined Dynamics: The product results in oscillations whose amplitude grows exponentially, characteristic of systems with damping or amplification effects.

Graphical Interpretation

Visualizing the solution reveals a waveform that oscillates with increasing amplitude over time. The peaks and troughs become more pronounced as t increases, highlighting the exponential growth component.

Methodology Summary and Critical Analysis

The process of solving this IVP involves several key steps:

1. Formulate the characteristic equation from the differential equation.

2. Solve the quadratic to find roots, determining the form of the general solution.

3. Construct the general solution based on the nature of roots (real or complex).

4. Apply initial conditions to solve for arbitrary constants.

5. Write the particular solution that satisfies both the differential equation and initial conditions.

This systematic approach is robust and widely applicable to similar linear differential equations.

Pros and Cons of the Method

Pros:

- Systematic and straightforward: Clear steps make it accessible for students and practitioners.

- Applicable to a broad class of equations: Particularly useful for constant coefficient linear ODEs.

- Provides explicit solutions: Facilitates qualitative analysis and visualization.

Cons:

- Limited to linear, constant coefficient equations: Not suitable for variable coefficient or nonlinear equations.

- Complex roots require additional interpretation: The oscillatory nature may be less intuitive in some contexts.

- Initial conditions must be compatible: Certain conditions may lead to no solution or require special handling.

Extensions and Related Techniques

While the method demonstrated is effective for this problem, other techniques can be employed for more complex scenarios:

- Laplace Transform: Useful for handling initial conditions and forcing functions.

- Series Solutions: Applicable when characteristic equations are difficult to solve analytically.

- Numerical Methods: Finite difference or Runge-Kutta methods for equations without closed-form solutions.

Applying the Laplace Transform

The Laplace transform converts differential equations into algebraic equations, simplifying the process especially when initial conditions are involved. For this problem, applying the Laplace transform yields a straightforward algebraic solution, which can be inverse-transformed to find Y(t).

Conclusion

Solving the initial-value problem Y" - 4y' + 8y = 0 with initial conditions Y(0) = 1 and Y'(0) = 2 demonstrates the power and elegance of analytical methods in differential equations. By deriving the characteristic roots, constructing the general solution, and applying initial conditions, we obtain a precise expression that describes the dynamic behavior of the system. The exponential oscillatory solution reflects a system with exponential growth coupled with sinusoidal oscillations—a scenario common in engineering, physics, and applied sciences. Understanding these methods enhances one's ability to analyze complex systems and develop intuition about their long-term behavior. While straightforward for constant coefficient linear equations, these techniques serve as a foundation for tackling more advanced and nonlinear problems, emphasizing the importance of mastering foundational mathematical tools.

Solve The Initial Value Problem Y 4y 8y 0 Y 0 1 Y 0 2

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