Use The Chain Rule To Find Dz/dt. Z = X2 + Y2 + Xy, X = Sin(t), Y = 4et

Use The Chain Rule To Find Dz/dt. $Z = X^2 + Y^2 + XY$, $X = \sin(t)$, $Y = 4e^t$

In this article, we will explore how to apply the chain rule to find the derivative of a composite function involving multiple variables. Specifically, we will analyze the function $Z = X^2 + Y^2 + XY$, where both X and Y are functions of a parameter t: $X = \sin(t)$ and $Y = 4e^t$. Our goal is to compute dz/dt, the rate of change of Z with respect to t, by carefully differentiating with respect to t using the multivariable chain rule.

Understanding the Problem and the Chain Rule

Multivariable Chain Rule Concept

The chain rule in multivariable calculus extends the familiar single-variable rule to functions of multiple variables. If a variable Z depends on variables X and Y, which themselves depend on t, then the derivative of Z with respect to t can be expressed as:

- **$dz/dt = (\partial Z/\partial X)(dX/dt) + (\partial Z/\partial Y)(dY/dt)$**

This formula captures how changes in t influence Z through their effects on X and Y. To compute dz/dt, we need three components:

1. Partial derivatives of Z with respect to X and Y: $\partial Z/\partial X$ and $\partial Z/\partial Y$
2. Derivatives of X and Y with respect to t: dX/dt and dY/dt
3. Combine all these to obtain dz/dt

Step 1: Find the partial derivatives of Z with respect to X and Y

Function Breakdown

The given function is:

$Z = X^2 + Y^2 + XY$

We need to compute:

- **$\partial Z/\partial X$**
- **$\partial Z/\partial Y$**

Calculating $\partial Z/\partial X$

Treating Y as a constant, differentiate Z with respect to X:

$\partial Z/\partial X = 2X + (Y) = 2X + Y$

Calculating $\partial Z/\partial Y$

Similarly, treating X as a constant, differentiate Z with respect to Y:

$\partial Z/\partial Y = 2Y + (X) = 2Y + X$

Step 2: Find the derivatives of X and Y with respect to t

Derivative of X = sin(t)

- **$dX/dt = \cos(t)$**

Derivative of Y = $4e^t$

- **$dY/dt = 4e^t$**

Step 3: Combine derivatives to find dz/dt

Applying the Chain Rule

Using:

$dz/dt = (\partial Z/\partial X)\ (dX/dt) + (\partial Z/\partial Y)\ (dY/dt)$

Substitute the derivatives obtained:

$dz/dt = (2X + Y)\ \cos(t) + (2Y + X)\ 4e^{t}$

Expressing everything in terms of t

Recall that:

$X = \sin(t)$

$Y = 4e^{t}$

Therefore, the derivative simplifies to:

$dz/dt = [2 \sin(t) + 4e^{t}]\ \cos(t) + [2\ 4e^{t} + \sin(t)]\ 4e^{t}$

Final Expression for dz/dt

Expand and Simplify

Distribute the terms:

$dz/dt = (2 \sin(t)\ \cos(t) + 4e^{t}\ \cos(t)) + (8e^{t}\ 4e^{t} + \sin(t)\ 4e^{t})$

Note that:

- *$8e^{t}\ 4e^{t} = 8\ 4\ e^{t}\ e^{t} = 32\ e^{2t}$*

Expressed Fully

$dz/dt = 2 \sin(t) \cos(t) + 4e^{t} \cos(t) + 32\ e^{2t} + 4\ e^{t} \sin(t)$

Conclusion and Summary

To find *dz/dt* for the function $Z = X^2 + Y^2 + XY$ with $X = \sin(t)$ and $Y = 4e^t$, we used the multivariable chain rule. The steps involved calculating partial derivatives of Z with respect to the variables X and Y, derivatives of X and Y with respect to t, and then substituting these into the chain rule formula. The final expression captures how Z changes with t, considering the nonlinear relationships between the variables and the parameter t.

Additional Notes and Applications

- This approach is fundamental in physics for understanding how compound systems evolve over time when multiple variables depend on a common parameter.
- Similarly, in engineering, the chain rule helps analyze dynamic systems where multiple interconnected components change simultaneously.
- Mastering the chain rule in multivariable calculus is essential for advanced topics like differential equations, optimization, and modeling real-world phenomena.

Summary of Key Steps

1. Identify the function and the variables involved
2. Calculate the partial derivatives of the function with respect to each variable
3. Find the derivatives of each variable with respect to t
4. Apply the chain rule formula to combine these derivatives
5. Simplify the resulting expression to get dz/dt

By following these steps carefully, you can extend this method to more complex functions involving multiple variables and parameters, making the chain rule an invaluable tool in calculus and applied mathematics.

Frequently Asked Questions

What is the function Z in terms of X and Y?

$Z = X^2 + Y^2 + XY$.

What are the given functions for X and Y in terms of t?

$X = \sin(t)$, $Y = 4e^t$.

How do you apply the chain rule to find dZ/dt for the given functions?

Use the chain rule: $dZ/dt = (\partial Z/\partial X)\, dX/dt + (\partial Z/\partial Y)\, dY/dt$.

What are the partial derivatives of Z with respect to X and Y?

$\partial Z/\partial X = 2X + Y$, and $\partial Z/\partial Y = 2Y + X$.

What are the derivatives of X and Y with respect to t?

$dX/dt = \cos(t)$, and $dY/dt = 4e^t$.

What is the final expression for dZ/dt?

$dZ/dt = (2X + Y) \cos(t) + (2Y + X)\, 4e^t$, where $X = \sin(t)$ and $Y = 4e^t$.

Additional Resources

Calculus: Mastering the Chain Rule to Find $\frac{dz}{dt}$ for $Z = X^2 + Y^2 + XY$

Introduction

In the realm of calculus, the chain rule stands as one of the most powerful tools for tackling derivatives of composite functions, especially when dealing with functions that depend on multiple variables, which in turn depend on a common parameter. Today, we delve into a classic yet intricate problem: finding $\frac{dz}{dt}$ where $Z = X^2 + Y^2 + XY$, with $X = \sin(t)$ and $Y = 4e^t$. This task exemplifies the elegance of the chain rule and its essential role in multivariable calculus.

Imagine you're exploring the dynamic world where variables evolve over time, and your goal is to understand how a combined quantity Z changes as time progresses. This is not only foundational in mathematical theory but also pivotal in fields like physics, engineering, and data science, where systems are often modeled with interconnected variables.

Setting the Stage: Understanding the Components

Before diving into derivatives, let's clarify each element involved:

- Variables X and Y: Both depend on t, meaning as t changes, so do $X(t)$ and $Y(t)$.
- $X(t) = \sin(t)$: a standard sine function, oscillating between -1 and 1.

- $Y(t) = 4e^t$: an exponential function scaled by 4, rapidly increasing as t grows.

- Function Z: A combination of X and Y, defined as:
$$
Z = X^2 + Y^2 + XY
$$
This function captures a quadratic relationship between the variables, intertwined with a product term.

The goal is to compute:
$$
\frac{dZ}{dt}
$$
which measures how Z changes over time, given the dependencies.

The Power of the Chain Rule in Multivariable Calculus

The chain rule extends beyond single-variable functions. When a function depends on multiple intermediate variables, which themselves depend on another variable (here, t), the derivative involves summing the partial derivatives appropriately:

$$
\frac{dZ}{dt} = \frac{\partial Z}{\partial X} \frac{dX}{dt} + \frac{\partial Z}{\partial Y} \frac{dY}{dt}
$$

Key points:

- $\frac{\partial Z}{\partial X}$ and $\frac{\partial Z}{\partial Y}$ are partial derivatives of Z with respect to X and Y.
- $\frac{dX}{dt}$ and $\frac{dY}{dt}$ are derivatives of X and Y with respect to t.

This formula embodies the multivariable chain rule, enabling us to differentiate composite functions efficiently.

Step-by-Step Solution Breakdown

Let's now walk through the process systematically.

1. Compute $\frac{\partial Z}{\partial X}$ and $\frac{\partial Z}{\partial Y}$

Recall:
$$
Z = X^2 + Y^2 + XY
$$

- Partial derivative with respect to X:

\[
\frac{\partial Z}{\partial X} = 2X + 0 + Y = 2X + Y
\]
Explanation:
- Derivative of \(X^2\) with respect to \(X\) is \(2X\).
- Derivative of \(Y^2\) with respect to \(X\) is 0 (since \(Y\) is treated as constant here).
- Derivative of \(XY\) with respect to \(X\) is \(Y\) (product rule simplifies to \(Y\) here as \(Y\) is constant relative to \(X\)).

- Partial derivative with respect to \(Y\):
\[
\frac{\partial Z}{\partial Y} = 0 + 2Y + X = 2Y + X
\]
Explanation:
- Derivative of \(X^2\) with respect to \(Y\) is 0.
- Derivative of \(Y^2\) with respect to \(Y\) is \(2Y\).
- Derivative of \(XY\) with respect to \(Y\) is \(X\).

2. Compute \(\frac{dX}{dt}\) and \(\frac{dY}{dt}\)

Given:
- \(X(t) = \sin(t)\)
- \(Y(t) = 4e^t\)

Calculate:

- \(\frac{dX}{dt}\):
\[
\frac{d}{dt} \sin(t) = \cos(t)
\]

- \(\frac{dY}{dt}\):
\[
\frac{d}{dt} 4e^t = 4e^t
\]

Synthesizing the Derivative \(\frac{dz}{dt}\)

Putting it all together:

\[
\frac{dz}{dt} = \left(2X + Y\right) \frac{dX}{dt} + \left(2Y + X\right) \frac{dY}{dt}
\]

Substitute the known functions:

\[
\boxed{
\frac{dz}{dt} = \left(2\sin(t) + 4e^t\right) \cos(t) + \left(2 \cdot 4e^t + \sin(t)\right) 4e^t

}
$$

Simplify step-by-step:

$$
\frac{dz}{dt} = \left(2\sin(t) + 4e^t\right) \cos(t) + \left(8e^t + \sin(t)\right) 4e^t
$$

Distribute:

$$
\frac{dz}{dt} = 2\sin(t) \cos(t) + 4e^t \cos(t) + 8e^t \cdot 4e^t + \sin(t) \cdot 4e^t
$$

Note that $8e^t \cdot 4e^t = 32 e^{2t}$ and $\sin(t) \cdot 4e^t = 4 e^t \sin(t)$.

Thus:

$$
\boxed{
\frac{dz}{dt} = 2 \sin(t) \cos(t) + 4 e^t \cos(t) + 32 e^{2t} + 4 e^t \sin(t)
}
$$

Final Expression and Interpretation

The derivative $\frac{dz}{dt}$ is expressed as a sum of four terms:

$$
\boxed{
\frac{dz}{dt} = 2 \sin(t) \cos(t) + 4 e^t \cos(t) + 32 e^{2t} + 4 e^t \sin(t)
}
$$

This formula encapsulates the combined effect of the changing $X(t)$ and $Y(t)$ on Z.

Practical Applications and Insights

- Oscillatory behavior: The terms involving $\sin(t)$ and $\cos(t)$ reflect oscillations from the sine function.
- Exponential growth: The e^{2t} term indicates exponential acceleration in the change, dominated by the exponential components.
- Composite effects: The mix of trigonometric and exponential functions demonstrates how different types of functions interact in dynamic systems.

Summary and Key Takeaways

- The chain rule extends elegantly to multivariable functions, requiring careful computation of partial derivatives and derivatives of the inner functions.
- For $Z = X^2 + Y^2 + XY$, the derivatives are straightforward once the partial derivatives are computed.
- Recognizing the nature of each component function (sine, exponential) simplifies the derivative calculations.
- The final derivative combines multiple effects, illustrating how variables influenced by different functions contribute to the overall change in Z.

Final Thoughts

Understanding how to use the chain rule in complex, multivariable contexts is essential for anyone seeking mastery in calculus. Whether modeling physical systems, analyzing signals, or solving engineering problems, this technique provides clarity and precision. The example explored here serves as a blueprint for tackling similar problems—highlighting the importance of methodical computation, attention to detail, and a solid grasp of fundamental derivatives.

By approaching each step with confidence, you enhance your mathematical toolkit and deepen your appreciation for the interconnectedness of functions—a true hallmark of advanced calculus.

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