32. Find A Homomorphism F From U(30) To U(30) With Kernel {1, 11} And F(7) 5 7.

32. Find A Homomorphism F From U(30) To U(30) With Kernel {1, 11} And F(7) 5 7.

Understanding the concept of group homomorphisms is fundamental in abstract algebra, particularly in the study of group structures and their mappings. This problem involves the multiplicative units modulo 30, denoted as U(30), which is the group of all integers less than 30 that are coprime to 30, under multiplication modulo 30. The task is to find a homomorphism $F: U(30) \to U(30)$ with a specific kernel and image condition: the kernel should be exactly $\{1, 11\}$, and the image of 7 under F should be 7 itself. This problem combines several key ideas—group kernels, images, and the structure of unit groups modulo composite numbers—and provides a rich example of homomorphism construction.

Understanding U(30): The Group of Units Modulo 30

Definition and Basic Properties

The group $U(30)$ consists of all integers a such that $1 \leq a < 30$ and $\gcd(a, 30) = 1$. The group operation is multiplication modulo 30. To understand the problem, we first examine the structure and size of $U(30)$.

- The elements of $U(30)$ are: $\{1, 7, 11, 13, 17, 19, 23, 29\}$.
- The order of $U(30)$ is $\varphi(30)$, where φ is Euler's totient function.
- Calculating $\varphi(30)$:

$$
\varphi(30) = 30 \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{5}\right) = 30 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 8.
$$

So, $|U(30)| = 8$.

- The group $U(30)$ is cyclic, as Euler's theorem suggests, and all elements generate various subgroups.

Group Homomorphisms and Kernels

What Is a Homomorphism?

A homomorphism $F: G \to H$ between groups satisfies:

$$
F(ab) = F(a)F(b) \quad \text{for all } a, b \in G.
$$

It preserves the group operation, and understanding its kernel and image provides insight into the structure of the groups involved.

Kernel of a Homomorphism

The kernel, $\ker(F)$, is the set of elements in G mapped to the identity in H. It is always a normal subgroup of G. In this problem, the kernel is specified as $\{1, 11\}$, a subgroup of $U(30)$.

- Since $\ker(F) = \{1, 11\}$, and the kernel is a subgroup, we verify whether this subgroup is valid and what it implies about F.

Constructing the Homomorphism F

Step 1: Validate the Kernel

The subgroup $\{1, 11\}$ must be a subgroup of $U(30)$. Let's verify:

- 1 and 11 are in $U(30)$.
- $11^2 \equiv 121 \equiv 1 \ (\text{mod } 30)$, so 11 has order 2.
- The subgroup $\{1, 11\}$ is cyclic of order 2, confirming it is a valid subgroup.

Since the kernel of a homomorphism is a normal subgroup, and $U(30)$ is abelian (hence all subgroups are normal), this is consistent.

Step 2: Use the First Isomorphism Theorem

The First Isomorphism Theorem states:

$$
U(30) / \ker(F) \cong \operatorname{Im}(F).
$$

Given $|\ker(F)| = 2$, and $|U(30)| = 8$, the quotient group has order:

$$
|U(30)/\ker(F)| = \frac{8}{2} = 4.
$$

Thus, the image of F is a subgroup of $U(30)$ of order 4.

Step 3: Determining Possible Images

Given the order constraints, the image of F must be a subgroup of $U(30)$ of order 4.

- Subgroups of $U(30)$ of order 4 are cyclic of order 4 or isomorphic to C_4.

- Elements of order dividing 4 in $U(30)$:
- 1 (order 1),
- Elements of order 2: 11,
- Elements of order 4: need to check.

Check the order of some elements:

- $7^2 \equiv 49 \equiv 19 \ (\text{mod } 30)$, not 1, so order > 2.
- $7^4 \equiv (7^2)^2 \equiv 19^2 \equiv 361 \equiv 1 \ (\text{mod } 30)$.

So, 7 has order 4:

$$
7^4 \equiv 1 \ (\text{mod } 30).
$$

Similarly, check the order of 13:

- $13^2 \equiv 169 \equiv 19$,
- $13^4 \equiv (13^2)^2 \equiv 19^2 \equiv 1$.

Thus, 13 also has order 4.

Designing F with the Given Conditions

Step 1: Fix $F(7)$

The problem states $F(7) = 7$. Since F is a homomorphism:

$$
F(7) = 7.
$$

This means that the element 7 maps to itself under F. Because 7 has order 4, and F is a homomorphism, the image of 7 must have order dividing 4. The options for $F(7)$ are:

- 7,
- 13,
- or possibly other elements of order dividing 4.

But since $F(7) = 7$, the homomorphism must send 7 to itself.

Step 2: Define F on generators

Since $U(30)$ is cyclic, generated by 7, defining F on 7 determines F entirely, provided the kernel condition is satisfied.

The key is to find a homomorphism F such that:

- $\ker(F) = \{1, 11\}$,
- $F(7) = 7$.

Because of the kernel, the subgroup $\{1, 11\}$ must be exactly the kernel, which is the preimage of the identity.

Constructing the Explicit Homomorphism F

Step 1: Use the quotient group structure

From the First Isomorphism Theorem:

$$
U(30) / \ker(F) \cong \operatorname{Im}(F),
$$

where $|\operatorname{Im}(F)| = 4$.

The quotient $U(30) / \{1, 11\}$ is of order 4. The elements of the quotient are cosets:

$$
\{ \{1, 11\}, \quad \{7, 19\}, \quad \{13, 23\}, \quad \{17, 29\} \}.
$$

Define F such that:

- $F(1) = 1$,
- $F(11) = 1$,
- $F(7) = 7$,
- $F(19) = F(7 \cdot 11) = F(7) \cdot F(11) = 7 \times 1 = 7$,
- $F(13)$ could be chosen as an element of order 4,
- $F(17)$ and $F(29)$ accordingly.

Step 2: Explicitly

Frequently Asked Questions

What is the group U(30) in the context of this problem?

U(30) is the multiplicative group of units modulo 30, consisting of all integers less than 30 that are coprime to 30, under multiplication modulo 30.

What does the kernel {1, 11} indicate about the homomorphism F from U(30) to U(30)?

The kernel {1, 11} indicates that these two elements are mapped to the identity element in the codomain, meaning F(1) = 1 and F(11) = 1, and that the kernel is a subgroup of U(30), specifically of order 2.

Is the set {1, 11} a subgroup of U(30)?

Yes, {1, 11} is a subgroup of U(30), as it contains the identity and is closed under multiplication modulo 30, with 11 being its own inverse.

Given F(7) = 5, how does this help in determining the homomorphism F?

Knowing F(7) = 5 provides a specific image for the element 7, which can be used along with the homomorphism properties to determine the images of other elements and ensure the homomorphism is well-defined.

How can we verify that F is a homomorphism from U(30) to U(30)?

To verify F is a homomorphism, we need to check that for all a, b in U(30), F(ab) ≡ F(a)F(b) (mod 30), and that the kernel is {1, 11} as specified.

What is the order of the subgroup {1, 11} in U(30)?

The subgroup {1, 11} has order 2, since it contains exactly two elements, and $11^2 \equiv 1 \pmod{30}$, confirming its structure as a subgroup of order 2.

Can we determine the entire homomorphism F based on the given information?

Yes, using the properties of group homomorphisms, the known value F(7) = 5, and the kernel, we can determine how F acts on generators and thus define the entire homomorphism.

What is the significance of the element 5 in the group U(30)?

The element 5 is a unit modulo 30, and since F(7) = 5, it indicates that 7 is mapped to 5 in the codomain, helping to establish the homomorphism's structure.

How does the kernel {1, 11} relate to the surjectivity of F?

The kernel {1, 11} being a subgroup of order 2 suggests that F is not necessarily injective, but the size of the image and the kernel's size imply F could be surjective onto a subgroup of U(30).

Additional Resources

32. Find A Homomorphism F From U(30) To U(30) With Kernel {1, 11} And F(7) ≠ 5

Introduction

In the realm of abstract algebra, especially group theory, the exploration of homomorphisms between groups holds a central role in understanding the structure and behavior of algebraic systems. The problem at hand—"Find a homomorphism $F: U(30) \to U(30)$ with kernel $\{1, 11\}$ and $F(7) \neq 5$"—serves as a fascinating case study illustrating the interplay between group structure, kernels, images, and the properties of units modulo a composite number.

This long-form investigation aims to dissect this problem thoroughly, offering insights into the structure of $U(30)$, the nature of homomorphisms, and the specific constraints posed. We will explore the necessary theoretical background, step through the problem-solving process meticulously, and evaluate the implications of the solution in the broader context of algebraic mappings.

Background and Theoretical Foundations

The Group $U(30)$: The Group of Units Modulo 30

The group $U(30)$ consists of all integers less than 30 that are coprime to 30, under multiplication modulo 30. Formally,

$$
U(30) = \{ a \in \mathbb{Z} \mid 1 \leq a < 30, \gcd(a, 30) = 1 \}
$$

The structure of $U(30)$ is well-understood through Euler's totient function:

$$
|U(30)| = \varphi(30) = 30 \times \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \left(1 - \frac{1}{5}\right) = 8
$$

Explicitly, the elements of $U(30)$ are:

$$
U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}
$$

This set forms a cyclic group of order 8 because 30 is square-free, and $U(n)$ is cyclic for $n = 2, 4, p^k, 2p^k$ with p an odd prime, but for $n=30$, the structure is known to be cyclic, or at least decomposable into cyclic components. In fact, $U(30) \cong C_2 \times C_4$, but in this case, it turns out to be cyclic of order 8 because

30's unit group is isomorphic to C_8.

To verify the structure, note that:

$$
U(30) \cong C_8
$$

which simplifies the analysis of homomorphisms.

Homomorphisms from a Cyclic Group

Any homomorphism $F: G \to G'$ where G is cyclic of order n and G' is any group, is determined entirely by the image of a generator g of G. Specifically, if $G = \langle g \rangle$, then

$$
F(g) \text{ can be any element in } G' \text{ with order dividing } n
$$

and

$$
\operatorname{Im}(F) = \langle F(g) \rangle
$$

The kernel of F is a subgroup of G, which, for a cyclic domain, is also cyclic and generated by g^{d}, where d divides n.

Problem Breakdown and Approach

The task is to find a homomorphism $F: U(30) \to U(30)$ such that:

- The kernel of F is $\{1, 11\}$
- $F(7) \neq 5$

Given the structure, we need to address several key points:

1. Kernel: Since the kernel is $\{1, 11\}$, a subgroup of $U(30)$, this subgroup must be normal (which all subgroups are in abelian groups) and of order 2.

2. Order of the Kernel: The kernel has 2 elements, so the homomorphism is a quotient of $U(30)$ by this kernel, which should be isomorphic to the image.

3. Image and Homomorphism Type: The First Isomorphism Theorem indicates

$$
U(30) / \ker F \cong \operatorname{Im} F \leq U(30)
$$

Since $|\ker F| = 2$, and $|U(30)| = 8$, the quotient has order 4, so the image $\operatorname{Im} F$ must be a subgroup of order 4.

Deep Dive: Constructing the Homomorphism

Step 1: Confirm the subgroup $\{1, 11\}$

Verify that $\{1, 11\}$ is a subgroup of $U(30)$. Since 1 is the identity, the key is to check if $11^2 \equiv 1 \pmod{30}$:

$$
11^2 = 121 \equiv 121 - 4 \times 30 = 121 - 120 = 1 \pmod{30}
$$

Yes, $11^2 \equiv 1 \pmod{30}$, so $\{1, 11\}$ is a cyclic subgroup of order 2 generated by 11.

Step 2: Determine the quotient group $U(30) / \{1, 11\}$

The quotient group has order:

$$
|U(30)| / |\{1, 11\}| = 8/2 = 4
$$

Thus, $U(30)/\{1, 11\}$ is a group of order 4, which is either cyclic or isomorphic to the Klein four-group V_4. Since $U(30)$ is cyclic of order 8, and the subgroup is of order 2, the quotient is cyclic of order 4:

\[

U(30)/\{1, 11\} \cong C_4
\]

Step 3: Homomorphism from a cyclic group

Given that, the homomorphism \(F \) corresponds to a surjective homomorphism from \(U(30) \) onto a subgroup of order 4, with kernel \(\{1, 11\} \).

The general form:

\[
F: U(30) \to U(30)
\]

with

\[
\ker F = \{1, 11\}
\]

and

\[
\operatorname{Im} F \cong U(30)/\ker F \cong C_4
\]

To construct such a homomorphism, choose a generator \(g \) of \(U(30) \), and define:

\[
F(g) = a
\]

where \(a \in U(30) \) has order dividing 4, i.e., order 1, 2, or 4.

Constructing \(F \): Explicit Steps

Step 4: Identify generators of \(U(30) \)

Let's verify the elements of \(U(30) \):

\[
U(30) = \{1, 7, 11, 13, 17, 19, 23, 29\}
\]

Determine the order of each:

- \(1 \): order 1
- \(7 \): Check \(7^2 \equiv 49 \equiv 19 \pmod{30} \), \(7^4 \equiv (7^2)^2 \equiv 19^2 \equiv 361 \equiv 1 \pmod{30} \). So, order divides 4; check if 7 has order 4:

\[
7^4 \equiv 1
\]

and no smaller positive power yields 1, so order 4.

Similarly, check others:

- \(11 \): order 2 (since \(11^2 \equiv 1 \))
- \(13 \): \(13^2 \equiv 169 \equiv 169 - 5 \times 30 = 169 - 150 = 19 \neq 1 \); \(13^4 \equiv

(13^2)^2 \equiv 19^2 \equiv 361 \equiv 1 \). So, order divides 4. Check:

\[
13^2 \equiv 19 \neq 1
\]

but

\[
13^4 \equiv 1
\]

and no smaller power yields 1, so order 4.

Similarly, for 17:

\[
17^2 = 289 \equiv 289

[32 Find A Homomorphism F From U 30 To U 30 With Kernel 1 11 And F 7 5 7]

32 Find A Homomorphism F From U 30 To U 30 With Kernel 1 11 And F 7 5 7

Related Articles

- The Infectious Substance Of Prions Is:a. Proteinb. Glycophosphatec. RNAd. DNA
- Once A _____ Is Installed, The Attacker Can Gain Full Access To The Computer.
- What Are Some New Things You Learned About The

Los Angeles Local Government?

Back to Home