4. (10 %) Find The Four Second Partial Derivatives Of The Function Z= Cos Xy.

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Understanding how to compute second partial derivatives of a function like $Z = \cos xy$ is fundamental in multivariable calculus. These derivatives reveal the curvature and concavity of the function in different directions, which are essential for analyzing the function's behavior, optimizing surfaces, and understanding saddle points or extrema. In this comprehensive guide, we will explore the process step-by-step, covering the calculation of all four second partial derivatives of $Z = \cos xy$, and highlight their significance in mathematical analysis.

Introduction to Partial Derivatives and Their Importance

Partial derivatives measure how a multivariable function changes as one variable varies, keeping other variables constant. For a function $Z = f(x,y)$, the first partial derivatives are:

- $\frac{\partial Z}{\partial x}$: rate of change of Z with respect to x, holding y constant.
- $\frac{\partial Z}{\partial y}$: rate of change of Z with respect to y, holding x constant.

Second partial derivatives extend this concept to how these rates themselves change, providing insights into the function's curvature:

- $\frac{\partial^2 Z}{\partial x^2}$
- $\frac{\partial^2 Z}{\partial y^2}$
- $\frac{\partial^2 Z}{\partial x \partial y}$
- $\frac{\partial^2 Z}{\partial y \partial x}$

For functions with continuous second derivatives, Clairaut's theorem states that mixed partial derivatives are equal:

$$
\frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial^2 Z}{\partial y \partial x}
$$

Thus, computing either mixed derivative suffices.

Understanding the Function \(Z = \cos xy \)

The function \(Z = \cos xy \) depends on the product of \(x \) and \(y \). Its behavior is governed by the interaction between these variables, making the derivatives particularly interesting because they involve applying the chain rule multiple times.

Key points about the function:

- The function oscillates between -1 and 1.
- Its shape depends on the values of \(x \) and \(y \).
- The derivatives involve products of \(x \), \(y \), and their derivatives.

Step-by-Step Calculation of the Second Partial Derivatives

Let's derive each of the four second partial derivatives systematically.

1. First Partial Derivative with Respect to \(x \): \(\frac{\partial Z}{\partial x} \)

Given:

\[
Z = \cos xy
\]

Applying the chain rule:

\[
\frac{\partial Z}{\partial x} = -\sin xy \times \frac{\partial (xy)}{\partial x}
\]

Since \(xy \) is a product:

\[
\frac{\partial (xy)}{\partial x} = y
\]

Thus:

\[
\boxed{
\frac{\partial Z}{\partial x} = - y \sin xy

}
\]

---

2. First Partial Derivative with Respect to \(y \): \(\frac{\partial Z}{\partial y} \)

Similarly:

\[
\frac{\partial Z}{\partial y} = - \sin xy \times \frac{\partial (xy)}{\partial y}
\]

Since:

\[
\frac{\partial (xy)}{\partial y} = x
\]

Therefore:

\[
\boxed{
\frac{\partial Z}{\partial y} = - x \sin xy
}
\]

---

3. Second Partial Derivative with Respect to \(x \): \(\frac{\partial^2 Z}{\partial x^2} \)

Now, differentiate \(\frac{\partial Z}{\partial x} = - y \sin xy \) with respect to \(x \):

\[
\frac{\partial^2 Z}{\partial x^2} = - y \times \frac{\partial (\sin xy)}{\partial x}
\]

Apply the chain rule again:

\[
\frac{\partial (\sin xy)}{\partial x} = \cos xy \times \frac{\partial (xy)}{\partial x} = \cos xy \times y
\]

Therefore:

$$
\frac{\partial^2 Z}{\partial x^2} = - y \times y \cos xy = - y^2 \cos xy
$$

---

4. Second Partial Derivative with Respect to y: $\frac{\partial^2 Z}{\partial y^2}$

Differentiate $\frac{\partial Z}{\partial y} = - x \sin xy$ with respect to y:

$$
\frac{\partial^2 Z}{\partial y^2} = - x \times \frac{\partial (\sin xy)}{\partial y}
$$

Applying the chain rule:

$$
\frac{\partial (\sin xy)}{\partial y} = \cos xy \times \frac{\partial (xy)}{\partial y} = \cos xy \times x
$$

Thus:

$$
\frac{\partial^2 Z}{\partial y^2} = - x \times x \cos xy = - x^2 \cos xy
$$

---

5. Mixed Second Partial Derivatives: $\frac{\partial^2 Z}{\partial x \partial y}$ and $\frac{\partial^2 Z}{\partial y \partial x}$

Let's compute $\frac{\partial^2 Z}{\partial x \partial y}$ by differentiating $\frac{\partial Z}{\partial x} = - y \sin xy$ with respect to y:

$$
\frac{\partial^2 Z}{\partial x \partial y} = - \left[\sin xy + y \times \cos xy \times x \right]
$$

Breaking it down:

- Differentiate $- y \sin xy$ with respect to y:

$$
\frac{\partial }{\partial y} \left(- y \sin xy \right) = - \left[\sin xy + y \times \cos xy \times x \right]

\]

So:

$$
\boxed{
\frac{\partial^2 Z}{\partial x \partial y} = - \sin xy - xy \cos xy
}
$$

Similarly, differentiating $\frac{\partial Z}{\partial y} = - x \sin xy$ with respect to x yields the same result, confirming Clairaut's theorem:

$$
\boxed{
\frac{\partial^2 Z}{\partial y \partial x} = - \sin xy - xy \cos xy
}
$$

Summary of All Second Partial Derivatives

Second Partial Derivative	Expression
$\frac{\partial^2 Z}{\partial x^2}$	$- y^2 \cos xy$
$\frac{\partial^2 Z}{\partial y^2}$	$- x^2 \cos xy$
$\frac{\partial^2 Z}{\partial x \partial y}$	$- \sin xy - xy \cos xy$
$\frac{\partial^2 Z}{\partial y \partial x}$	$- \sin xy - xy \cos xy$

Significance of the Second Partial Derivatives

Understanding these derivatives is critical in various applications:

- Analyzing Surface Curvature: The second derivatives help determine the concavity or convexity of the surface $Z = \cos xy$.
- Hessian Matrix Formation: These derivatives form the Hessian matrix, which is essential in optimization problems to identify local minima, maxima, or saddle points.
- Saddle Point Identification: For the function $Z = \cos xy$, the mixed derivatives indicate potential saddle points where the surface changes curvature.
- Modeling Physical Phenomena: In physics and engineering, such derivatives can describe wave behaviors, oscillations, or potential fields.

Conclusion

Calculating the second partial derivatives of $Z = \cos xy$ reveals intricate behaviors of the function in multiple directions. By systematically applying the chain rule and product rule, we derived all four second derivatives:

- $\frac{\partial^2 Z}{\partial x^2} = - y^2 \cos xy$
- $\frac{\partial^2 Z}{\partial y^2} = - x^2 \cos xy$
- $\frac{\partial^2 Z}{\partial x \partial y} = - \sin xy - xy \cos xy$
- $\frac{\partial^2 Z}{\partial y \partial x} = - \sin xy - xy \cos xy$

These derivatives are fundamental tools in multivariable calculus, enabling deeper analysis of the function's geometry and behavior. Whether you're a student, educator, or researcher, mastering the calculation of second

Frequently Asked Questions

What is the function for which we need to find the second partial derivatives?

The function is $Z = \cos(xy)$.

How do you find the first partial derivative of Z = cos(xy) with respect to x?

Using the chain rule, the first partial derivative with respect to x is $Z_x = -\sin(xy)\ y$.

What is the first partial derivative of Z = cos(xy) with respect to y?

Applying the chain rule, $Z_y = -\sin(xy)\ x$.

How do you compute the second partial derivative Z_xx?

Differentiate $Z_x = -y \sin(xy)$ with respect to x, applying the product rule: $Z_{xx} = -y [\cos(xy)\ y] = - y^2 \cos(xy)$.

What is the second partial derivative Z_yy?

Differentiate $Z_y = -x \sin(xy)$ with respect to y: $Z_{yy} = -x [\cos(xy)\ x] = - x^2 \cos(xy)$.

How do you find the mixed partial derivative Z_xy?

Differentiate $Z_x = - y \sin(xy)$ with respect to y: $Z_{xy} = - [\sin(xy) + y \cos(xy)\ x] = - \sin(xy) - xy \cos(xy)$.

Are the mixed partial derivatives Z_xy and Z_yx equal in this case?

Yes, because the function is continuously differentiable, so Z_xy = Z_yx, and both equal - sin(xy) - xy cos(xy).

What is the significance of finding these second partial derivatives?

They are essential in analyzing the curvature and local behavior of the function, such as determining concavity, saddle points, and applying the second derivative test.

Additional Resources

Partial Derivatives

Introduction to Partial Derivatives and the Function $Z = \cos xy$

Understanding the behavior of multivariable functions is fundamental in advanced calculus, engineering, physics, and many scientific disciplines. Central to this understanding are partial derivatives, which measure how a function changes with respect to one variable while keeping others constant. They are essential tools for analyzing functions of multiple variables, providing insights into slopes, rates of change, and the shape of surfaces.

Today, we focus on the function:
$$Z = \cos xy$$

This function combines both variables x and y within a cosine function, making it a rich candidate for exploration through partial derivatives. Our goal is to meticulously compute the second-order partial derivatives—specifically, the four mixed derivatives—and understand their significance.

Preliminary Concepts and Notation

Before diving into calculations, let's clarify some foundational concepts:

- Partial Derivative $\frac{\partial Z}{\partial x}$: The rate at which Z changes with respect to x, keeping y constant.
- Partial Derivative $\frac{\partial Z}{\partial y}$: The rate at which Z changes with respect to \(

y $), keeping x constant.
- Second Partial Derivatives: Derivatives of the first derivatives, which can be taken either with respect to the same variable or the other, leading to four types:
- $\frac{\partial^2 Z}{\partial x^2}$
- $\frac{\partial^2 Z}{\partial y^2}$
- $\frac{\partial^2 Z}{\partial x \partial y}$
- $\frac{\partial^2 Z}{\partial y \partial x}$

By Clairaut’s theorem (also known as Schwarz's theorem), if the function is sufficiently smooth, the mixed partial derivatives are equal:
$$ \frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial^2 Z}{\partial y \partial x} $$

Step-by-Step Calculation of the Second Partial Derivatives

Let's proceed systematically:

First, find $\frac{\partial Z}{\partial x}$

Given:
$$ Z = \cos xy $$

Applying the chain rule:
- The outer function is $\cos u$, with $u = xy$.
- Derivative of $\cos u$ w.r.t. u is $-\sin u$.
- Derivative of $u = xy$ w.r.t. x is y.

Therefore:
$$
\frac{\partial Z}{\partial x} = -\sin xy \times y = - y \sin xy
$$

Next, find $\frac{\partial Z}{\partial y}$

Similarly:
- Outer: $\cos u$
- $u = xy$, derivative w.r.t. y is x.

Thus:
$$
\frac{\partial Z}{\partial y} = - \sin xy \times x = - x \sin xy
$$

Second Partial Derivatives with Respect to the Same Variable

Now, differentiate the first derivatives again, respecting the variables.

1. Compute $\frac{\partial^2 Z}{\partial x^2}$

Start from:
$$
\frac{\partial Z}{\partial x} = - y \sin xy
$$

Applying the product rule:
$$
\frac{\partial^2 Z}{\partial x^2} = - y \frac{\partial}{\partial x} (\sin xy)
$$

Calculate $\frac{\partial}{\partial x} (\sin xy)$:

- Outer: $\sin u$, derivative: $\cos u$
- $u = xy$, derivative w.r.t. x: y

So:
$$
\frac{\partial}{\partial x} (\sin xy) = \cos xy \times y
$$

Putting it together:
$$
\frac{\partial^2 Z}{\partial x^2} = - y \times (\cos xy \times y) = - y^2 \cos xy
$$

---

2. Compute $\frac{\partial^2 Z}{\partial y^2}$

Start from:
$$
\frac{\partial Z}{\partial y} = - x \sin xy
$$

Differentiate w.r.t. y:
$$
\frac{\partial^2 Z}{\partial y^2} = - x \frac{\partial}{\partial y} (\sin xy)
$$

Calculate $\frac{\partial}{\partial y} (\sin xy)$:

- Outer: $\sin u$, derivative: $\cos u$
- $u = xy$, derivative w.r.t. y: x

Thus:

$$
\frac{\partial}{\partial y} (\sin xy) = \cos xy \times x
$$

Substitute:

$$
\frac{\partial^2 Z}{\partial y^2} = - x \times (\cos xy \times x) = - x^2 \cos xy
$$

Second Partial Derivatives with Respect to Different Variables (Mixed Derivatives)

3. Compute $\frac{\partial^2 Z}{\partial x \partial y}$

Starting from the first derivative w.r.t. x:

$$
\frac{\partial Z}{\partial x} = - y \sin xy
$$

Differentiate w.r.t. y:

$$
\frac{\partial^2 Z}{\partial x \partial y} = - y \frac{\partial}{\partial y} (\sin xy) - \sin xy \times \frac{\partial y}{\partial y}
$$

But note that the product rule applies here, since y is a variable:

$$
\frac{\partial}{\partial y} (- y \sin xy) = - \sin xy - y \frac{\partial}{\partial y} (\sin xy)
$$

Calculate $\frac{\partial}{\partial y} (\sin xy)$:

- Outer: $\sin u$
- $u = xy$, derivative w.r.t. y: x

Thus:

$$
\frac{\partial}{\partial y} (\sin xy) = \cos xy \times x
$$

Putting it together:

$$

$$
\frac{\partial^2 Z}{\partial x \partial y} = - \sin xy - y (\cos xy \times x) = - \sin xy - xy \cos xy
$$

4. Compute $\frac{\partial^2 Z}{\partial y \partial x}$

Starting from:

$$
\frac{\partial Z}{\partial y} = - x \sin xy
$$

Differentiate w.r.t. x:

$$
\frac{\partial^2 Z}{\partial y \partial x} = - \sin xy - x \frac{\partial}{\partial x} (\sin xy)
$$

Calculate $\frac{\partial}{\partial x} (\sin xy)$:

- Outer: $\sin u$, derivative: $\cos u$
- $u = xy$, derivative w.r.t. x: y

Hence:

$$
\frac{\partial}{\partial x} (\sin xy) = \cos xy \times y
$$

Plug into the expression:

$$
\frac{\partial^2 Z}{\partial y \partial x} = - \sin xy - x (\cos xy \times y) = - \sin xy - xy \cos xy
$$

Summary of All Second Partial Derivatives

Derivative Type	Expression	Explanation
$\frac{\partial^2 Z}{\partial x^2}$	$- y^2 \cos xy$	Change of Z w.r.t. x twice
$\frac{\partial^2 Z}{\partial y^2}$	$- x^2 \cos xy$	Change of Z w.r.t. y twice
$\frac{\partial^2 Z}{\partial x \partial y}$	$- \sin xy - xy \cos xy$	Mixed partial, differentiate $\frac{\partial Z}{\partial x}$ w.r.t. y
$\frac{\partial^2 Z}{\partial y \partial x}$	$- \sin xy - xy \cos xy$	Mixed partial, differentiate $\frac{\partial Z}{\partial y}$ w.r.t. x

Note that the mixed derivatives are equal, which confirms Clairaut’s theorem in this context.

Implications and Applications of These Derivatives

Having explicit formulas for these second derivatives offers powerful insights:

- Curvature Analysis: The second derivatives describe the concavity and convexity of the surface \(Z = \cos xy \).

[4 10 Find The Four Second Partial Derivatives Of The Function Z Cos Xy]

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