Calculate The Double Integral. R 4(1 X2) 1 Y2 Da, R = {(x, Y) | 0 X 4, 0 Y 1}

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Understanding how to evaluate double integrals is a fundamental skill in multivariable calculus, especially when dealing with functions over specific regions in the xy-plane. In this comprehensive guide, we will explore the process of calculating the double integral of the function $f(x, y) = 1 - x^2 + y^2$ over the region $R = \{(x, y) \mid 0 \leq x \leq 4,\, 0 \leq y \leq 1\}$. We will break down each step, provide detailed explanations, and include tips for optimizing your calculations. Whether you're a student learning calculus or a professional needing a refresher, this article will serve as an in-depth resource.

Understanding Double Integrals in Multivariable Calculus

Double integrals extend the concept of single-variable integrals to functions of two variables. They are used to compute areas, volumes, and other quantities where a function is integrated over a two-dimensional region.

Key points about double integrals:
- They are written in the form $\iint_R f(x, y) \, dA$, where R is a region in the xy-plane.
- The notation dA indicates a differential area element, which can be expressed as $dx\,dy$ or $dy\,dx$.
- The order of integration depends on the shape of the region R and the function involved.

Defining the Region R for the Integral

The problem specifies the region R as:

$$
R = \{(x, y) \mid 0 \leq x \leq 4,\; 0 \leq y \leq 1 \}
$$

This describes a rectangular region in the xy-plane with:
- x-values from 0 to 4
- y-values from 0 to 1

This simplicity allows for straightforward setup of the double integral.

Formulating the Double Integral

The double integral of the function over the region R is expressed as:

$$
\iint_R (1 - x^2 + y^2) \, dA
$$

Since R is a rectangle, the integral can be written as an iterated integral with limits corresponding to the region:

$$
\int_{x=0}^{4} \int_{y=0}^{1} (1 - x^2 + y^2) \, dy \, dx
$$

Alternatively, reversing the order:

$$
\int_{y=0}^{1} \int_{x=0}^{4} (1 - x^2 + y^2) \, dx \, dy
$$

Choosing the order depends on the ease of integration.

Step-by-Step Calculation of the Double Integral

Let's proceed with integrating with respect to y first, then x, which is often simpler due to the nature of the function.

Step 1: Set up the integral

$$
I = \int_{x=0}^{4} \left(\int_{y=0}^{1} (1 - x^2 + y^2) \, dy \right) dx
$$

Step 2: Integrate with respect to y

For a fixed x, integrate:

$$
\int_{0}^{1} (1 - x^2 + y^2) \, dy
$$

Break it into parts:

$$
\int_{0}^{1} (1 - x^2) \, dy + \int_{0}^{1} y^2 \, dy
$$

Since $(1 - x^2)$ is constant with respect to y:

$$
(1 - x^2) \int_{0}^{1} dy + \int_{0}^{1} y^2 \, dy
$$

Calculate each:

- $\int_{0}^{1} dy = 1$
- $\int_{0}^{1} y^2 \, dy = \left[\frac{y^3}{3} \right]_0^1 = \frac{1}{3}$

Thus:

$$
(1 - x^2)(1) + \frac{1}{3} = 1 - x^2 + \frac{1}{3} = \left(1 + \frac{1}{3}\right) - x^2 = \frac{4}{3} - x^2
$$

Step 3: Integrate with respect to x

Now, the integral reduces to:

$$
I = \int_{0}^{4} \left(\frac{4}{3} - x^2 \right) dx
$$

Calculate:

$$
I = \int_{0}^{4} \frac{4}{3} \, dx - \int_{0}^{4} x^2 \, dx
$$

Compute each integral:

- $\int_{0}^{4} \frac{4}{3} \, dx = \frac{4}{3} \times 4 = \frac{16}{3}$
- $\int_{0}^{4} x^2 \, dx = \left[\frac{x^3}{3} \right]_0^4 = \frac{4^3}{3} = \frac{64}{3}$

Step 4: Combine the results

Subtract:

$$
I = \frac{16}{3} - \frac{64}{3} = - \frac{48}{3} = -16
$$

Final Result and Interpretation

The value of the double integral over the specified region is:

$$
\boxed{
\iint_R (1 - x^2 + y^2) \, dA = -16
}
$$

This negative result indicates that, over the region, the function's negative parts dominate, leading to an overall negative accumulated value.

Tips for Computing Double Integrals Effectively

To optimize your calculations of double integrals, consider the following tips:

1. Choose the Most Convenient Order of Integration
- Analyze the integrand and region to decide whether integrating with respect to x or y first simplifies the process.

2. Break Down Complex Integrals
- Split the integrand into simpler components that are easier to integrate separately.

3. Visualize the Region
- Sketch the region R to better understand the limits and possible substitutions.

4. Use Symmetry When Possible
- Symmetric regions or functions can simplify calculations, especially when integrating over symmetric bounds.

5. Remember Standard Integrals
- Memorize common integrals such as $\int y^n dy$, which speeds up calculations.

Applications of Double Integrals in Real-World Scenarios

Double integrals are essential in various fields, including:

- Physics: Calculating mass, center of mass, and moments of inertia.
- Engineering: Determining the load distribution over a surface.
- Economics: Computing total revenue or cost over a geographic region.
- Environmental Science: Estimating pollutant concentrations over an area.

Understanding how to evaluate double integrals like $\iint_R (1 - x^2 + y^2) \, dA$ enables professionals to analyze complex systems involving two-variable functions efficiently.

Summary and Key Takeaways

- To compute the double integral $\iint_R (1 - x^2 + y^2) \, dA$ over $R = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 1 \}$, set up the iterated integral based on the region's limits.
- Integrate first with respect to y, then x, or vice versa, depending on the integrand and region.
- The calculated value of the integral in this example is -16.
- Use strategic techniques and visualization to simplify complex double integrals.

By mastering the process of calculating double integrals over rectangular regions, you'll enhance your problem-solving skills in multivariable calculus and expand your ability to analyze real-world phenomena involving two-variable functions. Whether you're studying academic coursework or applying these techniques professionally, understanding the step-by-step approach outlined here will serve as a valuable foundation.

Frequently Asked Questions

What is the region R for the double integral ∫∫_R (4(1 - x^2) - y^2) dA?

The region R is defined by $0 \leq x \leq 4$ and $0 \leq y \leq 1$, forming a rectangle in the xy-plane.

How do you set up the double integral over the region R for the given function?

Since R is a rectangle, the integral is set up as $\int_0^4 \int_0^1 (4(1 - x^2) - y^2)\, dy\, dx$.

What is the first step to evaluate the double integral $\int_0^4 \int_0^1 (4(1 - x^2) - y^2)\, dy\, dx$?

Begin by integrating the inner integral with respect to y while treating x as a constant.

How do you compute the inner integral $\int_0^1 (4(1 - x^2) - y^2)\, dy$?

Integrate term-by-term: $\int_0^1 4(1 - x^2)\, dy - \int_0^1 y^2\, dy$, resulting in $4(1 - x^2)\, y \,|_0^1 - (1/3)\, y^3 \,|_0^1 = 4(1 - x^2) - 1/3$.

After computing the inner integral, what is the resulting integral to evaluate?

The outer integral becomes $\int_0^4 [4(1 - x^2) - 1/3]\, dx$.

How do you evaluate the outer integral $\int_0^4 [4(1 - x^2) - 1/3]\, dx$?

Split the integral: $\int_0^4 4(1 - x^2)\, dx - \int_0^4 (1/3)\, dx$. Integrate each separately and then combine the results.

What is the value of the outer integral after computation?

Calculating: $\int_0^4 4(1 - x^2)\, dx = 4[x - (x^3)/3]\_0^4 = 4(4 - (64/3)) = 4(4 -$

21.333...) = 4(−17.333...) = −69.333...; and $\int_0^4$ (1/3) dx = (1/3)4 = 4/3. So, the total is −69.333... − 1.333... = −70.666..., which simplifies to −212/3.

What is the final value of the double integral ∫∫_R (4(1 − x^2) − y^2) dA?

The value of the double integral is −212/3.

Additional Resources

Calculate The Double Integral. R 4(1 X2) 1 Y2 Da, R = {(x, Y) | 0 X 4, 0 Y 1} is a fundamental problem in multivariable calculus that involves evaluating a double integral over a specified region. This type of problem is critical in various fields such as physics, engineering, and economics, where it is necessary to compute areas, volumes, or other quantities that depend on two variables. The process of calculating double integrals allows us to understand the accumulation of a quantity over a two-dimensional area, providing insight into complex systems and spatial phenomena. In this article, we will thoroughly analyze this particular integral, breaking down the problem, exploring methods to evaluate it, and discussing its applications and implications.

Understanding the Problem and Its Components

Defining the Region R

The given region, R, is specified as:

$R = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 1\}$

This describes a rectangular region in the xy-plane, bounded by the lines x = 0, x = 4, y = 0, and y = 1. Visualizing this region helps in setting up the double integral correctly. It is straightforward, which simplifies the integration process, especially when the integrand is a product of functions in x and y.

Understanding the Integrand

The integrand is given as:

$f(x, y) = (1 - x^2)\, y^2$

This function combines a quadratic term in x, $(1 - x^2)$, with a quadratic term in y, y^2. Its behavior varies across the region R, and understanding its structure is key in choosing the method of integration. Because the integrand is separable into functions of x and y, it offers the advantage of straightforward integration when set up properly.

Setting Up the Double Integral

Order of Integration

For this problem, since the region R is a rectangle, the order of integration can be chosen freely. Typically, the double integral is written as:

$\iint_R (1 - x^2) y^2 \, dA$

which can be expressed as either:

- integrating with respect to y first, then x:

$\int_0^4 \int_0^1 (1 - x^2) y^2 \, dy \, dx$

or

- integrating with respect to x first, then y:

$\int_0^1 \int_0^4 (1 - x^2) y^2 \, dx \, dy$

Choosing the order that simplifies calculations is often advantageous. In this case, integrating with respect to y first simplifies because the integrand separates nicely.

Expressing the Integral Explicitly

Using the order of integration where y is integrated first:

$\iint_R (1 - x^2) y^2 \, dA = \int_0^4 [\int_0^1 (1 - x^2) y^2 \, dy] \, dx$

This setup clearly indicates the limits for y and x, allowing us to proceed with the integration step-by-step.

Step-by-Step Evaluation of the Double Integral

Integrating with Respect to y

Focus on the inner integral:

$\int_0^1 (1 - x^2) y^2 \, dy$

Since $(1 - x^2)$ is treated as a constant with respect to y, it can be factored out:

$(1 - x^2) \int_0^1 y^2 \, dy$

The integral of y^2 from 0 to 1 is straightforward:

$\int_0^1 y^2 \, dy = [y^3 / 3]_0^1 = 1/3$

Thus, the inner integral simplifies to:

$(1 - x^2) (1/3)$

Integrating with Respect to x

Now, the double integral reduces to:

$\int_0^4 (1 - x^2) (1/3) \, dx$

Factor out 1/3:

$(1/3) \int_0^4 (1 - x^2) \, dx$

Compute the integral:

$\int_0^4 1 \, dx - \int_0^4 x^2 \, dx$

which is:

$[x]_0^4 - [x^3 / 3]_0^4$

Calculating these:

- x from 0 to 4: 4 - 0 = 4
- $x^3 / 3$ from 0 to 4: (64 / 3) - 0 = 64/3

Putting it all together:

$(1/3) (4 - 64/3) = (1/3) ((12/3) - (64/3)) = (1/3) (-52/3) = -52 / 9$

Final Result:

The value of the double integral is −52/9.

Analysis of the Result and Its Significance

Interpretation of the Result

The negative value of the double integral indicates that, over the region R, the integrand's negative contributions outweigh the positive ones. Since $(1 - x^2)$ becomes negative for $x > 1$ and up to 2, and is positive for $x < 1$, the overall integral reflects the balance of these regions weighted by y^2. The fact that the integral yields a negative result suggests that the dominant negative contributions from $x > 1$ significantly influence the total.

Implications in Applications

In practical contexts, such as physics or engineering, the double integral could represent quantities like net flux, accumulated mass, or other physical phenomena where negative values might signify direction or imbalance. Understanding the sign and magnitude of such integrals is crucial for interpreting real-world data.

Features, Pros, and Cons of the Calculation Method

Features:

- The integrand's separability simplifies the integration process.
- The rectangular region allows straightforward setting of limits.
- Step-by-step integration provides clarity and minimizes errors.

Pros:

- Ease of computation due to simple limits and separable functions.
- The method generalizes well to other rectangular regions and similar integrands.
- Provides exact analytical solution, valuable for theoretical and practical purposes.

Cons:

- Limited to regions with simple geometry; complex shapes require more advanced techniques.
- The integral's negative result might be counterintuitive without proper interpretation.
- Assumes familiarity with basic integral calculus; more advanced regions or integrands might require coordinate transformations or numerical methods.

Extensions and Variations

This problem can be extended or varied in several ways:

- Changing the region R to a non-rectangular shape, necessitating different limits or coordinate transformations.
- Using different integrands that involve exponential, trigonometric, or other functions.
- Evaluating triple integrals for volumetric or higher-dimensional problems.
- Applying numerical methods like Monte Carlo integration or Simpson's rule for complicated regions or integrands.

Conclusion

Calculating the double integral of $(1 - x^2) y^2$ over the region $R = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq 1\}$ illustrates fundamental principles of multivariable calculus, including setting up integrals, choosing the order of integration, and executing straightforward calculations. The process underscores the importance of understanding the region and the integrand's structure, which simplifies the evaluation process. The final result, $-52/9$, not only provides a precise numerical value but also offers insights into the nature of the integrand's behavior over the specified region. Such calculations are essential tools in various scientific disciplines, enabling practitioners to quantify and analyze complex systems with confidence and precision.

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