Expand The Complex Function F(z) = 1/z(z-2) In A Laurent Series About Z = 0.

Expand The Complex Function F(z) = 1/z(z-2) In A Laurent Series About Z = 0. In complex analysis, the Laurent series provides a powerful tool for representing functions with singularities within an annulus. When dealing with rational functions such as F(z) = 1 / [z(z - 2)], it becomes essential to understand how to expand these functions into Laurent series around specific points—in this case, at z = 0. This allows us to analyze the behavior of the function near the singularity and understand its structure in terms of principal and regular parts. In this article, we will explore the process of expanding F(z) into a Laurent series about z = 0, including the necessary steps, the regions of convergence, and the implications of such an expansion.

Understanding the Function and Its Singularities

Before diving into the expansion process, it is crucial to grasp the nature of the function and its singularities, as they dictate the structure and the annulus within which the Laurent series converges.

Function Overview

The function F(z) = 1 / [z(z - 2)] is a rational function with two singularities:
- A simple pole at z = 0
- A simple pole at z = 2

These singularities influence the possible Laurent expansions around different points. Since our focus is on z = 0, the key is to examine the behavior of F(z) in neighborhoods around z = 0, excluding the other singularity at z = 2.

Singularities and Regions of Convergence

The Laurent series expansion around z = 0 will have different forms depending on the annular region considered:
- Inner Region (|z| < 2): The series converges inside the circle of radius 2, excluding the singularity at 0.
- Outer Region (|z| > 0): The series can be expanded outside the circle, but here, since we're focused on z = 0, the primary interest is in the inner region, where |z| < 2.

The goal is to find the Laurent expansion valid in the annulus $0 < |z| < 2$.

Expressing F(z) in a Suitable Form for Expansion

To find the Laurent series centered at $z = 0$, we need to manipulate $F(z)$ into a form that separates terms into power series expansions valid in the desired region.

Partial Fraction Decomposition

The first step is to decompose $F(z)$:

$$
F(z) = \frac{1}{z(z - 2)} = \frac{A}{z} + \frac{B}{z - 2}
$$

To find A and B, multiply both sides by $z(z - 2)$:

$$
1 = A(z - 2) + Bz
$$

Set $z = 0$:

$$
1 = A(0 - 2) + B \times 0 \Rightarrow 1 = -2A \Rightarrow A = -\frac{1}{2}
$$

Set $z = 2$:

$$
1 = A(2 - 2) + B \times 2 \Rightarrow 1 = 0 + 2B \Rightarrow B = \frac{1}{2}
$$

Thus,

$$
F(z) = -\frac{1}{2z} + \frac{1}{2(z - 2)}
$$

Now, the expansion around $z = 0$ can be written as:

$$
F(z) = -\frac{1}{2z} + \frac{1}{2(z - 2)}
$$

The first term is already in a Laurent form with a pole at $z=0$. The second term requires further expansion to express it as a Laurent (or power) series valid for $|z| < 2$.

Expanding the Second Term: $\frac{1}{2(z-2)}$

To develop the Laurent series, we focus on expanding $\frac{1}{z - 2}$ as a series around $z = 0$, valid for $|z| < 2$.

Rearrangement for Geometric Series

Note that:

$$
\frac{1}{z - 2} = -\frac{1}{2} \times \frac{1}{1 - \frac{z}{2}}
$$

Provided that $|z/2| < 1$, i.e., $|z| < 2$, we can expand:

$$
\frac{1}{1 - \frac{z}{2}} = \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n
$$

Therefore:

$$
\frac{1}{z - 2} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}
$$

Multiplying by the factor 1/2 from the partial fraction decomposition:

$$
\frac{1}{2(z - 2)} = \frac{1}{2} \times \left(-\frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n}\right) = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{z^n}{2^n}
$$

Now, the Laurent expansion of $F(z)$ around $z=0$, valid for $|z|<2$, is:

$$
F(z) = -\frac{1}{2z} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{z^n}{2^n}
$$

Expressed explicitly:

$$
F(z) = -\frac{1}{2z} - \frac{1}{4} \left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots \right)
$$

which simplifies to:

$$
F(z) = -\frac{1}{2z} - \frac{1}{4} - \frac{z}{4 \times 2} - \frac{z^2}{4 \times 2^2} - \frac{z^3}{4 \times 2^3} - \cdots
$$

or

$$
F(z) = -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots
$$

This Laurent series converges for all z such that $|z| < 2$.

Summary of the Laurent Expansion

The Laurent series expansion of $F(z) = 1 / [z(z - 2)]$ around $z = 0$ in the annulus $0 < |z| < 2$ is:

$$
\boxed{
F(z) = -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots
}
$$

- The principal part (terms with negative powers) is only the $-\frac{1}{2z}$ term.
- The regular part (non-negative powers) is the infinite series starting from the constant term onward.

This expansion captures the behavior of the function near $z=0$, excluding the singularity at the origin itself.

Implications and Applications of the Laurent Series

Understanding the Laurent expansion of $F(z)$ around $z=0$ provides insights into several aspects of complex analysis:

- **Residue Calculation:** The coefficient of the $\frac{1}{z}$ term in the Laurent series is the residue at $z=0$. Here, the residue is $-\frac{1}{2}$, which is crucial for evaluating contour integrals around $z=0$.

- **Singularity Classification:** The presence of a simple pole at $z=0$ is evident from the Laurent series. The principal part's leading term confirms this classification.

- **Analytic Continuation:** The Laurent series provides a way to extend the function analytically within the annulus of convergence, excluding singularities.

Conclusion

Expanding the function $F(z) = 1 / [z(z - 2)]$ into a Laurent series centered at $z=0$ involves partial fraction decomposition, geometric series expansion, and careful consideration of the annular region of convergence. The resulting series:

$$
F(z) = -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots
$$

Frequently Asked Questions

What is the Laurent series expansion of the function F(z) = 1 / [z(z - 2)] around z = 0?

The Laurent series expansion of $F(z)$ around $z = 0$ is $F(z) = -1/2 (1/z) - 1/4 - (z/8) - (z^2/16) - \ldots$, valid for $|z| < 2$.

How do you find the Laurent series of F(z) = 1 / [z(z - 2)] around z = 0?

By decomposing $F(z)$ into partial fractions: $1/[z(z - 2)] = A/z + B/(z - 2)$, then expanding each term as a power series around $z = 0$, and summing the series accordingly.

What are the partial fraction coefficients for F(z) = 1 / [z(z - 2)]?

Solving $1 = A(z - 2) + Bz$, setting $z = 0$ gives $A = -1/2$, and $z = 2$ gives $B = 1/2$. Thus, the partial fractions are $(-1/2)/z + (1/2)/(z - 2)$.

For which values of z does the Laurent series expansion of F(z) converge?

The Laurent series around $z = 0$ converges for $|z| < 2$, since the expansion of $1/(z - 2)$ as a geometric series converges in that disk.

How can I expand 1/(z - 2) into a power series around z = 0?

Express $1/(z - 2)$ as $-1/2 \; 1 / (1 - z/2)$, then expand as a geometric series: $-1/2 \sum_{n=0}^{\infty} (z/2)^n = -1/2 - z/4 - z^2/8 - \dots$, valid for $|z| < 2$.

What is the significance of the singularity at z=0 for this Laurent expansion?

Since $z=0$ is a pole of order 1 for $F(z)$, the Laurent series will include terms with negative powers of z, specifically a $(1/z)$ term, reflecting the singularity.

Can the Laurent series be used to compute residues of F(z) at z=0?

Yes, the coefficient of $1/z$ in the Laurent series expansion around $z=0$ is the residue at $z=0$, which in this case is $-1/2$.

How does the Laurent expansion around z=0 compare to the expansion around z=2?

The Laurent expansion around $z=0$ focuses on the behavior near the origin, including the pole at $z=0$, while expansion around $z=2$ would involve a different series valid outside the circle $|z - 2| > 2$, reflecting the different singularity.

What is the importance of understanding Laurent series in complex analysis?

Laurent series provide a way to analyze functions near singularities, compute residues, evaluate integrals via residue theorem, and understand the local behavior of complex functions.

Additional Resources

Expand The Complex Function $F(z) = 1/z(z-2)$ In A Laurent Series About $Z = 0$

Understanding complex functions and their series expansions is a cornerstone of complex analysis, a field that elegantly blends algebra, calculus, and geometry. Among the many functions studied, rational functions—those expressed as ratios of polynomials—are particularly significant due to their simplicity and widespread applications. Today, we explore a classic example: the function $F(z) = \frac{1}{z(z-2)}$. Our goal is to expand this function into a Laurent series centered at $z = 0$, shedding light on its behavior near this singularity.

This task not only provides insights into the local properties of the function but also exemplifies the broader techniques used to handle singularities, residues, and series expansions in complex analysis. Whether you're a student, a researcher, or an enthusiast eager to deepen your understanding, this comprehensive exploration will guide you through the process step-by-step, illustrating key concepts and methods.

Understanding the Function and Its Singularities

Before diving into the series expansion, it's crucial to understand the nature of $F(z)$.

The Function $F(z) = \frac{1}{z(z-2)}$

The function is a rational expression with two simple poles: one at $z=0$ and another at $z=2$. These points are where the function becomes unbounded:

- At $z=0$: The denominator vanishes due to the factor z, leading to a pole.
- At $z=2$: The denominator vanishes due to the factor $z-2$, leading to a second pole.

Our focus is on expanding $F(z)$ into a Laurent series about $z=0$, which involves expressing $F(z)$ as a series that may include negative powers of z. This kind of expansion is valid in an annulus (ring-shaped region) around $z=0$, specifically where the series converges.

The Rationale for Laurent Series Expansion at $z=0$

Why a Laurent Series?

In complex analysis, a Taylor series provides a power series expansion valid within a disk centered at the expansion point, assuming the function is analytic there. However, when the function has singularities—points where it is not analytic—Taylor series may not suffice. Instead, the Laurent series, which allows for negative powers, becomes a powerful tool to describe the behavior of the function near singularities.

The Laurent Series Region of Validity

The Laurent series expansion centered at $z=0$ depends on the location of other singularities:

- If the point $z=2$ (the other pole) is outside the annulus, the series converges in $0 < |z| < R$, where R is less than the distance to the next singularity.
- Since $z=2$ is at a distance of 2 from 0, the natural annulus for expansion is $0 < |z| < 2$.

Our task is to find the Laurent series of $F(z)$ valid in this annulus.

Decomposition of $F(z)$ Using Partial Fractions

A common approach to expand rational functions is to decompose them into partial fractions. This simplifies the process of series expansion, especially when dealing with negative powers.

Step 1: Partial Fraction Decomposition

Express $F(z)$ as a sum of simpler fractions:

$$
F(z) = \frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}
$$

To find A and B, multiply both sides by $z(z-2)$:

$$
1 = A(z-2) + Bz
$$

Set $z=0$:

$$
1 = A(0-2) + B \times 0 \Rightarrow 1 = -2A \Rightarrow A = -\frac{1}{2}
$$

Set $z=2$:

$$
1 = A(2-2) + B \times 2 \Rightarrow 1 = 0 + 2B \Rightarrow B = \frac{1}{2}
$$

Result of Partial Fractions:

$$
F(z) = -\frac{1}{2z} + \frac{1}{2(z-2)}
$$

This decomposition enables us to handle each term separately, especially the second term, which contains the $z-2$ denominator.

Series Expansion of Each Term

Now, we focus on expanding each term into a Laurent series centered at $z=0$.

Term 1: $-\frac{1}{2z}$

This is already in a simple Laurent form:

$$
-\frac{1}{2z}
$$

It is valid in the entire annulus $0 < |z| < \infty$, except at $z=0$, where it has a pole.

Term 2: $\frac{1}{2(z-2)}$

This term requires more work, as it involves $z-2$, which is not centered at 0. To expand it into a Laurent series about $z=0$, we need to manipulate it into a form suitable for series expansion.

Expanding $\frac{1}{z-2}$ Around $z=0$

Step 1: Rewrite the denominator

Express $\frac{1}{z-2}$ as:

$$
\frac{1}{z-2} = -\frac{1}{2 - z}
$$

Now, the goal is to find a series expansion of $\frac{1}{2 - z}$ around $z=0$.

Step 2: Recognize the geometric series form

Recall that for $|z| < 2$:

$$
\frac{1}{2 - z} = \frac{1}{2} \times \frac{1}{1 - \frac{z}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n
$$

This is a geometric series expansion valid when $|z/2| < 1$, i.e., $|z| < 2$.

Step 3: Write the expansion

Therefore,

$$
\frac{1}{z-2} = - \frac{1}{2} \sum_{n=0}^\infty \left(\frac{z}{2}\right)^n = - \sum_{n=0}^\infty \frac{z^n}{2^{n+1}}
$$

Multiplying by $\frac{1}{2}$ (from earlier):

$$
\frac{1}{2(z-2)} = \frac{1}{2} \times \left(- \sum_{n=0}^\infty \frac{z^n}{2^{n+1}} \right) = - \sum_{n=0}^\infty \frac{z^n}{2^{n+2}}
$$

Combining the Series Components

Now, the entire Laurent series for $F(z)$ around $z=0$ in the annulus $0 < |z| < 2$ is:

$$
F(z) = -\frac{1}{2z} - \sum_{n=0}^\infty \frac{z^n}{2^{n+2}}
$$

Expressed explicitly:

$$
F(z) = -\frac{1}{2z} - \frac{1}{2^2} - \frac{z}{2^3} - \frac{z^2}{2^4} - \frac{z^3}{2^5} - \cdots
$$

or:

$$
\boxed{
F(z) = -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots
}
$$

Validity and Region of Convergence

This Laurent series converges for all z such that $0 < |z| < 2$, excluding the singularity at $z=0$ (due to the pole) and the singularity at $z=2$ (which defines the outer boundary of the annulus).

Interpretation of the Laurent Series

The Laurent expansion provides a detailed local picture of the function's behavior near $z=0$:

- The principal part: $-\frac{1}{2z}$, which indicates a simple pole at $z=0$.
- The regular part: the power series $- \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \cdots$, which is analytic at $z=0$.

This expansion is invaluable for computing residues, evaluating integrals via the residue theorem, and analyzing the function's local properties.

Broader Implications and Applications

Residue Calculation

The residue of $F(z)$ at $z=0$

Expand The Complex Function F Z 1 Z Z 2 In A Laurent Series About Z 0

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