Find The Area Of The Surface Generated By Revolvingy=x3/9,0x2around The X-axis

Find The Area Of The Surface Generated By Revolving $y = x^3/9$, 0 □ x □ 2 around the X-axis

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Introduction to Surface of Revolution

In calculus, the concept of a surface of revolution involves revolving a curve around an axis to generate a three-dimensional surface. This technique is fundamental in understanding the shape and surface area of solids formed by such revolutions. Calculating the surface area of these solids requires a specific integral formulation, often involving derivatives of the curve function and the radius of revolution.

In this article, we aim to determine the surface area of the surface generated when the curve defined by the function $y = x^3/9$, for x in the interval [0, 2], is revolved around the x-axis. This problem encompasses key calculus concepts, including derivatives, integrals, and the application of the surface area formula for revolution.

Understanding the Problem

The Given Function

The function provided is:

$$ y = \frac{x^3}{9} $$

for:

$$ 0 \leq x \leq 2 $$

This is a cubic function scaled vertically by a factor of 1/9, starting at the origin (0,0) and increasing as x increases.

The Axis of Revolution

The curve is revolved around the x-axis, which runs horizontally along y=0. This axis acts as the axis of symmetry and the axis around which the surface will be generated.

Goal

Calculate the surface area of the resulting surface of revolution, which can be visualized as a "shell" formed when the curve is spun around the x-axis.

Mathematical Foundations for Surface Area Calculation

Surface Area of Revolution Formula

For a curve y = f(x), revolved around the x-axis from x = a to x = b, the surface area S is given by:

$$
S = 2 \pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
$$

Where:

- $y = f(x)$ is the function defining the curve.

- $\frac{dy}{dx}$ is the derivative of y with respect to x.
- a, b are the bounds of x over which the curve is revolved.

Why this formula?

- The term $2\pi y$ represents the circumference of a typical "slice" of the surface at position x.
- The differential element of surface area is this circumference multiplied by the differential arc length $ds = \sqrt{1 + (dy/dx)^2} \, dx$.

Step-by-Step Calculation of Surface Area

Step 1: Define the function and its derivative

Given:

$$
y = \frac{x^3}{9}
$$

Calculate the derivative:

$$
\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x^3}{9} \right) = \frac{3x^2}{9} = \frac{x^2}{3}
$$

Step 2: Set up the integral

Substitute y and dy/dx into the surface area formula:

$$
S = 2 \pi \int_0^2 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
$$

which becomes:

$$
S = 2 \pi \int_0^2 \left(\frac{x^3}{9} \right) \sqrt{1 + \left(\frac{x^2}{3} \right)^2} \, dx
$$

Simplify inside the square root:

$$
1 + \left(\frac{x^2}{3} \right)^2 = 1 + \frac{x^4}{9}
$$

Step 3: Write the integral explicitly

$$
S = 2 \pi \int_0^2 \frac{x^3}{9} \sqrt{1 + \frac{x^4}{9}} \, dx
$$

Factor out constants:

$$
S = \frac{2 \pi}{9} \int_0^2 x^3 \sqrt{1 + \frac{x^4}{9}} \, dx
$$

Step 4: Simplify the integrand

Express the square root as:

$$
\sqrt{1 + \frac{x^4}{9}} = \sqrt{\frac{9 + x^4}{9}} = \frac{\sqrt{9 + x^4}}{3}
$$

Substitute back:

$$
S = \frac{2 \pi}{9} \int_0^2 x^3 \cdot \frac{\sqrt{9 + x^4}}{3} \, dx = \frac{2 \pi}{27} \int_0^2 x^3 \sqrt{9 + x^4} \, dx
$$

Thus, the surface area integral simplifies to:

$$
S = \frac{2 \pi}{27} \int_0^2 x^3 \sqrt{9 + x^4} \, dx
$$

Evaluating the Integral

Step 5: Substitution method

To evaluate:

$$
I = \int_0^2 x^3 \sqrt{9 + x^4} \, dx
$$

we notice the expression under the root involves x^4. Let's consider substitution:

$$
t = x^2 \Rightarrow dt = 2x \, dx \Rightarrow x \, dx = \frac{dt}{2}
$$

Express x^3 and x^4:

$$
x^3 = x \cdot x^2 = x \cdot t
$$

Rewrite the integral:

$$
I = \int_{t=0}^{t=4} x^3 \sqrt{9 + x^4} \, dx
$$

But since $x^2 = t$, then $x = \sqrt{t}$, and:

$$
x^3 = (\sqrt{t})^3 = t^{3/2}
$$

and:

$$
x^4 = (x^2)^2 = t^2
$$

Now, express dx:

\[

$$
dt = 2x\, dx \Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2 \sqrt{t}}
$$

Substitute into the integral:

$$
I = \int_{0}^{4} t^{3/2} \sqrt{9 + t^2} \cdot \frac{dt}{2 \sqrt{t}} = \frac{1}{2} \int_{0}^{4} t^{3/2} \cdot \frac{\sqrt{9 + t^2}}{\sqrt{t}} \, dt
$$

Simplify $t^{3/2} / \sqrt{t} = t^{3/2} / t^{1/2} = t^{(3/2) - (1/2)} = t^{1} = t$:

$$
I = \frac{1}{2} \int_{0}^{4} t \sqrt{9 + t^2} \, dt
$$

Step 6: Final substitution

The integral reduces to:

$$
I = \frac{1}{2} \int_0^4 t \sqrt{9 + t^2} \, dt
$$

This is a standard integral suitable for substitution:

$$
u = 9 + t^2 \Rightarrow du = 2 t \, dt \Rightarrow t \, dt = \frac{du}{2}
$$

Express the integral in terms of u:

$$
I = \frac{1}{2} \int_{u=9}^{u=25} \sqrt{u} \cdot \frac{du}{2} = \frac{1}{4} \int_9^{25} u^{1/2} \, du
$$

Now, integrate:

$$
\int u^{1/2} \, du = \frac{2}{3} u^{3/2}
$$

So:

$$
I = \frac{1}{4} \times \frac{2}{3} \left[u^{3/2} \right]_9^{25} = \frac{1}{6} \left[25^{3/2} - 9^{3/2} \right]
$$

Calculate each term:

$$
25^{3/2} = (25^{1/2})^{3} = 5^{3} = 125
$$

$$
9^{3/2} = (9^{1/2})^{3} = 3^{3} = 27
$$

Therefore:

$$
I = \frac{1}{6} (125 - 27) = \frac{1}{6} \times 98 = \frac{98}{6} = \frac{49}{3}
$$

Final Surface Area Calculation

Recall that:

\[

S = \frac{2 \pi}{27} \times I

\]

Substitute \(I = \frac{49}{3} \):

\[

S = \frac{2 \pi}{27

Frequently Asked Questions

What is the general formula for finding the surface area generated by revolving a curve y = f(x) around the x-axis?

The surface area S is given by the integral $S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$, where [a, b] is the interval of x values.

How do you identify the limits of integration for the surface area of $y = x^3/9$ from x = 0 to x = 2?

The limits are determined by the given interval for x; in this case, from x = 0 to x = 2, so the integral will be evaluated over [0, 2].

What is the derivative of $y = x^3/9$, and how is it used in calculating the surface area?

The derivative $y' = d/dx\ (x^3/9) = (3x^2)/9 = x^2/3$. It is used to compute $(1 + y'^2)$ inside the integral for surface area calculation.

How do you set up the integral to find the surface area generated by revolving $y = x^3/9$ around the x-axis from 0 to 2?

Use the formula $S = 2\square\ \square\_0\text{^}2\ y(x)\ \square(1 + y'(x)^2)\ dx$. Substituting $y = x^3/9$ and $y' = x^2/3$, the integral becomes $S = 2\square\ \square\_0\text{^}2\ (x^3/9)\ \square(1 + (x^2/3)^2)\ dx$.

How can the integral for the surface area be simplified before evaluation?

Simplify the expression inside the square root: $1 + (x\square/9) = (9 + x\square)/9$. Then, $\square(1 + y'^2) = \square((9 + x\square)/9) = \square(9 + x\square)/3$. The integral becomes $S = (2\square/3)\ \square\_0\text{^}2\ x^3\ \square(9 + x\square)\ dx$.

What substitution can be used to evaluate the integral for the surface area involving $\square(9 + x\square)$?

A substitution such as $t = x^2$ can be used, leading to $dt = 2x\ dx$, which simplifies the integral to a form involving $\square(9 + t^2)$.

What is the significance of understanding the derivative y' in calculating the surface area?

The derivative y' captures the slope of the curve, which affects the surface area calculation by accounting for the curve's steepness and ensuring accurate surface measurement.

Can this method be applied to any function y = f(x) for revolving around the x-axis?

Yes, the method applies generally; for any smooth function y = f(x), the surface area generated by revolution around the x-axis is given by $S = 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2}\, dx$, provided the function is integrable over [a, b].

What are common challenges faced when calculating the surface area of a revolved curve like $y = x^3/9$?

Challenges include setting up the correct integral, simplifying complex radicals, choosing appropriate substitutions, and accurately evaluating the resulting integral, especially when involving higher-degree polynomials or radicals.

Additional Resources

Find The Area Of The Surface Generated By Revolving $y = x^3/9$, $0 \leq x \leq 2$, around the X-axis

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Introduction

In the realm of calculus, the exploration of surface areas generated by revolving curves around an axis is a fundamental yet intriguing topic that combines geometric intuition with analytical rigor. Specifically, calculating the surface area of a surface of revolution provides insights into the shape and extent of objects formed by such processes, which has applications in engineering, physics, and even manufacturing. This article delves deeply into the problem of finding the surface area generated by revolving the curve $y = \frac{x^3}{9}$ over the interval $[0, 2]$ around the x-axis, unraveling the mathematical principles, step-by-step procedures, and the significance of the result.

Understanding the Problem: Surface of Revolution

What is a Surface of Revolution?

A surface of revolution is a three-dimensional surface obtained by rotating a two-dimensional curve about an axis. In our case, the curve $y = \frac{x^3}{9}$ is revolved around the x-axis, producing a three-dimensional surface resembling a symmetrical shell or bowl.

Why Focus on the Surface Area?

While volume calculations tell us how much space an object encloses, surface area calculations inform us about the extent of the material needed to create the surface, which is crucial in manufacturing, material science, and architecture.

Mathematical Foundations

To find the surface area generated by revolving a curve $y = f(x)$ over an interval $[a, b]$ around the x-axis, the standard formula is:

$$
S = 2\pi \int_{a}^{b} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
$$

Key Components:

- y: The function value at each point, representing the radius of the revolution.
- $\sqrt{1 + (dy/dx)^2}$: The differential arc length element, accounting for the curve's slope.

Step-by-Step Approach

1. Define the function and interval

Given:

$$
y = \frac{x^3}{9}
$$

with $x \in [0, 2]$.

2. Calculate the derivative dy/dx

Differentiating:

$$
dy/dx = \frac{d}{dx} \left(\frac{x^3}{9} \right) = \frac{3x^2}{9} = \frac{x^2}{3}
$$

3. Compute $1 + (dy/dx)^2$

$$
1 + \left(\frac{x^2}{3} \right)^2 = 1 + \frac{x^4}{9} = \frac{9 + x^4}{9}
$$

4. Set up the integral for surface area

Plugging into the formula:

$$
S = 2\pi \int_{0}^{2} y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

Substitute y and the expression under the root:

$$
S = 2\pi \int_0^2 \left(\frac{x^3}{9} \right) \sqrt{ \frac{9 + x^4}{9} } \, dx
$$

Simplify the square root:

$$
\sqrt{ \frac{9 + x^4}{9} } = \frac{\sqrt{9 + x^4}}{3}
$$

Thus, the integrand becomes:

$$
\left(\frac{x^3}{9} \right) \times \left(\frac{\sqrt{9 + x^4}}{3} \right) = \frac{x^3 \sqrt{9 + x^4}}{27}
$$

The surface area integral simplifies to:

$$
S = 2\pi \int_0^2 \frac{x^3 \sqrt{9 + x^4}}{27} \, dx
$$

Factor out constants:

$$
S = \frac{2\pi}{27} \int_0^2 x^3 \sqrt{9 + x^4} \, dx
$$

Analytical Evaluation of the Integral

The integral:

\[

I = \int_0^2 x^3 \sqrt{9 + x^4} \, dx

\]

requires substitution for a feasible solution.

1. Substitution Strategy

Observe the expression inside the square root:

\[

u = 9 + x^4

\]

then:

\[

du = 4x^3 \, dx

\]

which directly relates to the \(x^3 dx \) term present in the integral.

Express \(x^3 dx \):

\[

x^3 dx = \frac{1}{4} du

\]

When \(x = 0 \):

\[

u = 9 + 0 = 9

\]

When \(x = 2 \):

\[

u = 9 + (2)^4 = 9 + 16 = 25

\]

Rearranged, the integral becomes:

\[
I = \int_{u=9}^{25} \sqrt{u} \times \frac{1}{4} du
\]

which simplifies to:

\[
I = \frac{1}{4} \int_9^{25} u^{1/2} \, du
\]

2. Computing the Integral

The integral:

\[
\int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C
\]

Applying limits:

\[
I = \frac{1}{4} \times \frac{2}{3} \left[u^{3/2} \right]_{9}^{25} = \frac{1}{6} \left(25^{3/2} - 9^{3/2} \right)
\]

Calculate each term:

- \(25^{3/2} \):

\[
25^{3/2} = (25^{1/2})^3 = (5)^3 = 125
\]

- \(9^{3/2} \):

\[
9^{3/2} = (9^{1/2})^3 = (3)^3 = 27
\]

Therefore:
\[
I = \frac{1}{6} (125 - 27) = \frac{1}{6} \times 98 = \frac{98}{6} = \frac{49}{3}
\]

Final Surface Area Calculation

Recall the earlier constants:
\[
S = \frac{2\pi}{27} \times I
\]
Substitute \(I = \frac{49}{3} \):
\[
S = \frac{2\pi}{27} \times \frac{49}{3} = \frac{2\pi \times 49}{27 \times 3} = \frac{98\pi}{81}
\]

Result

\[
\boxed{
\textbf{Surface Area} = \frac{98\pi}{81}

}

\]

This is the exact value of the surface area generated by revolving the curve $y = \frac{x^3}{9}$ over $[0, 2]$ around the x-axis.

Significance and Applications

Mathematical Significance:

This calculation exemplifies the power of substitution techniques and the application of the surface area formula for revolution—concepts central to integral calculus. It demonstrates how complex-looking integrals can often be reduced to manageable forms using strategic substitutions, leading to exact solutions.

Practical Applications:

- Manufacturing: Designing objects like bowls, vases, or shells where surface material estimation is crucial.
- Physics and Engineering: Analyzing objects with rotational symmetry, such as turbines or pipes, where surface characteristics influence thermal, aerodynamic, or structural properties.
- Mathematical Modeling: Understanding the geometric properties of surfaces generated by revolution helps in computer graphics, CAD modeling, and scientific visualization.

Additional Insights and Extensions

- Numerical Approximation: For more complex functions or when exact expressions are intractable, numerical integration methods such as Simpson's rule or Gaussian quadrature are valuable.

- Surface Area of Other Curves: Similar techniques can extend to other functions, including exponential, trigonometric, or polynomial curves.
- Volume vs. Surface Area: While this article focused on surface area, the volume of the solid of revolution can be calculated using the disk method, with the formula:

$$
V = \pi \int_{a}^{b} y^2 \, dx
$$

- Parametric and Alternative Methods: For complex curves, parametrization or computer algebra systems facilitate calculations.

Conclusion

The calculation of the surface area generated by revolving the curve $y = \frac{x^3}{9}$ around the x-axis over $[0, 2]$ underscores the elegance and utility of integral calculus in geometric problems. By applying substitution techniques and understanding the geometric interpretation of derivatives and integrals, one can derive precise and insightful results. The exact surface area, \(\frac{98\pi

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