Find The Gradient Field Of The Function, F(x, Y, Z) = (3x^2 + 2y^2 + Z^2)^-1/2.

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Understanding how to compute the gradient field of a multivariable function is fundamental in vector calculus, with applications spanning physics, engineering, and mathematics. In this article, we will explore the process step-by-step to find the gradient field of the given function:

$$ F(x, y, z) = \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{1}{2}} $$

This comprehensive guide aims to clarify the concepts involved, provide detailed calculations, and discuss the significance of the gradient field in various contexts.

Introduction to Gradient Fields

Before diving into the specific function, it is essential to understand what a gradient field is and its role in vector calculus.

What is a Gradient Field?

The gradient field of a scalar function $F(x, y, z)$ is a vector field that assigns to each point in space a vector pointing in the direction of the greatest rate of increase of F. Mathematically, it is represented as:

$$ \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) $$

where:

- $\frac{\partial F}{\partial x}$ is the partial derivative of F with respect to x,
- $\frac{\partial F}{\partial y}$ is the partial derivative of F with respect to y,
- $\frac{\partial F}{\partial z}$ is the partial derivative of F with respect to z.

Significance of the Gradient Field

Understanding the gradient field provides insights into:

- Direction and magnitude of the steepest ascent,
- Critical points where the gradient is zero (maxima, minima, saddle points),
- Flow lines or lines of steepest increase in physical phenomena such as heat, fluid flow, or potential fields.

Step-by-Step Calculation of the Gradient Field for F(x, y, z)

Given the function:

$$ F(x, y, z) = \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{1}{2}} $$

our goal is to compute ∇F, i.e., the vector of partial derivatives.

Step 1: Recognize the Structure of the Function

The function is composed of an inner quadratic form:

$$ G(x, y, z) = 3x^2 + 2y^2 + z^2 $$

and an outer power:

$$ F(x, y, z) = G(x, y, z)^{-\frac{1}{2}} $$

Applying the chain rule will be effective here.

Step 2: Compute the Partial Derivatives

We will compute each of the partial derivatives separately.

Partial derivative with respect to x

Given $F = G^{-\frac{1}{2}}$, and $G = 3x^2 + 2y^2 + z^2$,

$$
\frac{\partial F}{\partial x} = \frac{d}{dx} \left(G^{-\frac{1}{2}} \right) = -\frac{1}{2} G^{-\frac{3}{2}} \cdot \frac{\partial G}{\partial x}
$$

Since $\frac{\partial G}{\partial x} = 6x$, we have:

$$
\frac{\partial F}{\partial x} = -\frac{1}{2} \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}} \cdot 6x = -3x \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}}
$$

\]

Partial derivative with respect to y

Similarly,

$$
\frac{\partial F}{\partial y} = -\frac{1}{2} G^{-\frac{3}{2}} \cdot \frac{\partial G}{\partial y}
$$

where $\frac{\partial G}{\partial y} = 4y$, so:

$$
\frac{\partial F}{\partial y} = -\frac{1}{2} \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}} \cdot 4y = -2y \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}}
$$

Partial derivative with respect to z

And,

$$
\frac{\partial F}{\partial z} = -\frac{1}{2} G^{-\frac{3}{2}} \cdot \frac{\partial G}{\partial z}
$$

with $\frac{\partial G}{\partial z} = 2z$, thus:

$$
\frac{\partial F}{\partial z} = -\frac{1}{2} \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}} \cdot 2z = -z \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}}
$$

Final Expression of the Gradient Field

Combining all the partial derivatives, the gradient field $\nabla F(x, y, z)$ is:

$$
\nabla F(x, y, z) = \left(-3x \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}}, \quad -2y \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}}, \quad -z \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}} \right)

\]

This vector field describes the direction and magnitude of the steepest increase of the function F at any point (x, y, z).

Interpreting the Gradient Field

Understanding the physical and geometric significance of this gradient field involves analyzing its properties.

Direction of the Gradient

At any point (x, y, z):

- The gradient points in the direction of the greatest increase of F.
- Since all components are negative multiples of the coordinates, the gradient points toward the origin (0,0,0), indicating that F decreases as one moves away from the origin.

Magnitude of the Gradient

The magnitude of the gradient vector is:

$$
|\nabla F| = \sqrt{\left(-3x (\cdots) \right)^2 + \left(-2y (\cdots) \right)^2 + \left(-z (\cdots) \right)^2}
$$

which simplifies to:

$$
|\nabla F| = \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{3}{2}} \sqrt{9x^2 + 4y^2 + z^2}
$$

This expression indicates how rapidly F increases in the direction of the gradient at each point.

Applications of the Gradient Field

The gradient field derived above has various practical applications:

- Potential Theory: The function resembles a potential function in physics, where the gradient represents force fields such as gravity or electrostatics.

- Optimization: Identifying critical points (where the gradient is zero) helps locate maxima, minima, or saddle points.
- Flow and Field Visualization: Visualizing the gradient vectors helps understand how a quantity changes in space, important in fluid dynamics and electromagnetic field analysis.
- Surface and Level Set Analysis: The gradient is perpendicular to level surfaces of F. Here, level surfaces are given by:

$$3x^2 + 2y^2 + z^2 = C$$

for constant C. The gradient vectors are normal to these surfaces, aiding in geometric understanding.

Conclusion

Finding the gradient field of the function $F(x, y, z) = (3x^2 + 2y^2 + z^2)^{-\frac{1}{2}}$ involves applying the chain rule to the inner quadratic form and recognizing the structure of the function. The resulting vector field:

\[
\nabla F(x, y, z) = \left(-3x (3x^2 + 2y^2 + z^2)^{-\frac{3}{2}}, \quad -2y (3x^2 + 2y^

Frequently Asked Questions

What is the general method to find the gradient field of a scalar function like F(x, y, z)?

To find the gradient field of a scalar function, you compute its partial derivatives with respect to each variable and form the vector of these derivatives. For F(x, y, z), the gradient is $\nabla F = (\partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$.

How do you compute the gradient of $F(x, y, z) = (3x^2 + 2y^2 + z^2)^{-1/2}$?

Apply the chain rule: first find the gradient of the inner function $G(x, y, z) = 3x^2 + 2y^2 + z^2$, then multiply by the derivative of the outer function with respect to G, which is $-1/2\, G^{(-3/2)}$.

What is the gradient of the inner function $G(x, y, z) = 3x^2 + 2y^2 + z^2$?

The gradient of G is $(\partial G/\partial x, \partial G/\partial y, \partial G/\partial z) = (6x, 4y, 2z)$.

How do you express the gradient of F(x, y, z) in terms of G and its derivatives?

Using the chain rule, $\nabla F = -1/2\ G^{-3/2}\ \nabla G$, which simplifies to $\nabla F = -1/2\ (3x^2 + 2y^2 + z^2)^{-3/2}\ (6x, 4y, 2z)$.

What is the simplified form of the gradient ∇F(x, y, z) for the given function?

The gradient simplifies to $(\nabla F)_x = -3x / (3x^2 + 2y^2 + z^2)^{3/2}$, $(\nabla F)_y = -2y / (3x^2 + 2y^2 + z^2)^{3/2}$, and $(\nabla F)_z = -z / (3x^2 + 2y^2 + z^2)^{3/2}$.

Why is understanding the gradient field important in multivariable calculus?

The gradient field indicates the direction and rate of fastest increase of the function, which is essential in optimization, physics (like force fields), and understanding the behavior of multivariable functions.

Can the gradient field be used to find level surfaces of the function? If yes, how?

Yes. The gradient is perpendicular to level surfaces. To find level surfaces, set F(x, y, z) equal to a constant, and the gradient helps understand the orientation of these surfaces.

What is the significance of the negative sign in the gradient of F(x, y, z)?

The negative sign indicates that the gradient points in the direction of decreasing values of the function, which is useful in gradient descent algorithms and understanding the flow of the field.

How does the homogeneity degree of the function F(x, y, z) affect its gradient field?

Since F is a negative fractional power of a quadratic form, it is homogeneous of degree -1. This affects the scaling behavior of its gradient, which also exhibits specific homogeneity properties, influencing how the gradient changes under scaling transformations.

Additional Resources

Find The Gradient Field Of The Function, $F(x, y, z) = (3x^2 + 2y^2 + z^2)^{-1/2}$

Understanding the gradient field of a multivariable function is a fundamental aspect of vector calculus, with applications ranging from physics to engineering and beyond. In this detailed exploration, we focus on the function $F(x, y, z) = (3x^2 + 2y^2 + z^2)^{-\frac{1}{2}}$, guiding readers through the process of finding its gradient field step-by-step. This comprehensive review aims to clarify the concepts, provide clear methodology, and highlight key features of the gradient field associated with this specific function.

Introduction to the Gradient Field

The gradient field of a scalar function is a vector field that points in the direction of the greatest rate of increase of the function at each point. It encapsulates vital information about the function's behavior in space, such as identifying local maxima, minima, and saddle points. Mathematically, for a function $F(x, y, z)$, the gradient vector ∇F is composed of its partial derivatives:

$$
\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)
$$

Finding the gradient field involves calculating these derivatives and expressing the result as a vector function of (x, y, z). For the given function $F(x, y, z) = (3x^2 + 2y^2 + z^2)^{-\frac{1}{2}}$, this process reveals the nature of how the function varies throughout three-dimensional space.

Understanding the Function $F(x, y, z)$

Before diving into derivatives, it’s crucial to analyze the structure of $F(x, y, z)$. The function is a composition of a quadratic form inside a negative fractional power:

$$
F(x, y, z) = \left(3x^2 + 2y^2 + z^2 \right)^{-\frac{1}{2}}
$$

This resembles a scaled Euclidean norm, where the quadratic form $Q(x, y, z) = 3x^2 + 2y^2 + z^2$ defines an ellipsoidal shape in space. Since the function involves the inverse square root, it exhibits a decay as the quadratic form increases, approaching zero at infinity, and tending towards infinity as the quadratic form approaches zero.

Features of $F(x, y, z)$:

- Radial symmetry around the origin, scaled differently along each axis due to the coefficients.
- Singularity at the origin: as $Q \to 0$, $F \to \infty$.
- Decreases smoothly away from the origin.

Calculating the Gradient of $F(x, y, z)$

The gradient ∇F involves partial derivatives with respect to each variable. Because F is a composition, applying the chain rule is essential.

Step 1: Rewrite the function for clarity

Let:

$$
Q(x, y, z) = 3x^2 + 2y^2 + z^2
$$

Then,

$$
F(x, y, z) = Q(x, y, z)^{-\frac{1}{2}}
$$

Step 2: Compute partial derivatives of F

Using the chain rule:

$$
\frac{\partial F}{\partial x} = -\frac{1}{2} Q^{-\frac{3}{2}} \cdot \frac{\partial Q}{\partial x}
$$

Similarly for y and z:

$$
\frac{\partial F}{\partial y} = -\frac{1}{2} Q^{-\frac{3}{2}} \cdot \frac{\partial Q}{\partial y}
$$

$$
\frac{\partial F}{\partial z} = -\frac{1}{2} Q^{-\frac{3}{2}} \cdot \frac{\partial Q}{\partial z}
$$

Step 3: Derivatives of Q

Since $Q = 3x^2 + 2y^2 + z^2$, we have:

$$
\frac{\partial Q}{\partial x} = 6x
$$

$$
\frac{\partial Q}{\partial y} = 4y
$$

$$
\frac{\partial Q}{\partial z} = 2z
$$

Step 4: Final expressions for the gradient components

Plugging derivatives into the chain rule expressions:

$$
\frac{\partial F}{\partial x} = -\frac{1}{2} Q^{-\frac{3}{2}} \cdot 6x = -3x \cdot Q^{-\frac{3}{2}}
$$

$$
\frac{\partial F}{\partial y} = -\frac{1}{2} Q^{-\frac{3}{2}} \cdot 4y = -2y \cdot Q^{-\frac{3}{2}}
$$

$$
\frac{\partial F}{\partial z} = -\frac{1}{2} Q^{-\frac{3}{2}} \cdot 2z = -z \cdot Q^{-\frac{3}{2}}
$$

Expressing the Gradient Field

The gradient vector field ∇F can now be written as:

$$
\nabla F(x, y, z) = \left(-3x \cdot Q^{-\frac{3}{2}},\ -2y \cdot Q^{-\frac{3}{2}},\ -z \cdot Q^{-\frac{3}{2}} \right)
$$

Or, factoring out $Q^{-\frac{3}{2}}$:

$$
\nabla F(x, y, z) = -Q^{-\frac{3}{2}} \left(3x,\ 2y,\ z \right)
$$

where

$$
Q = 3x^2 + 2y^2 + z^2
$$

This compact form clearly indicates the direction and magnitude of the gradient at any point (x, y, z).

Analysis of the Gradient Field

Having explicitly computed the gradient, it's insightful to analyze its properties:

Directionality

- The gradient points in the direction of the steepest ascent of F. Since F decreases as Q increases, the gradient vector points toward the origin, indicating the function attains its maximum at the origin (where $F \to \infty$) and diminishes outward.

Magnitude

- The magnitude of the gradient:

$$
|\nabla F| = Q^{-\frac{3}{2}} \sqrt{(3x)^2 + (2y)^2 + z^2} = Q^{-\frac{3}{2}} \sqrt{9x^2 + 4y^2 + z^2}
$$

- Near the origin, the gradient becomes unbounded (due to $Q \to 0$), which aligns with the singularity at the origin.

Critical Points

- The only critical point occurs where $\nabla F = 0$, which implies:

$$
-Q^{-\frac{3}{2}} (3x, 2y, z) = 0
$$

- Since $Q > 0$ away from the origin, the only solution is at:

$$
(0, 0, 0)

\]

- This point is a singularity; the gradient is not defined exactly at the origin, but considering limits, it indicates a maximum of \(F\).

Applications and Significance

Understanding the gradient field of \(F(x, y, z)\) offers insights into various physical and mathematical phenomena, including:

- Potential fields: The gradient often corresponds to force fields derived from potential functions.
- Level surfaces: The surfaces of constant \(F\) are ellipsoids, and the gradient vectors are perpendicular to these surfaces.
- Optimization: The gradient points toward local maxima or minima, crucial for optimization algorithms.

Pros and Cons of the Gradient Field Analysis

Pros:

- Provides a clear visualization of the function's behavior in space.
- Assists in identifying critical points and behavior at infinity.
- Useful in physics for understanding force fields and potential gradients.

Cons:

- Singularities at the origin require careful handling.
- The negative power complicates certain interpretations, especially near zero.
- The gradient field is undefined exactly at the point where the quadratic form vanishes.

Features Summary

- The gradient field is proportional to \(\left(3x, 2y, z \right)\), scaled by \(Q^{-\frac{3}{2}}\).
- It points inward toward the origin, indicating \(F\) has a local maximum at the

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Find The Gradient Field Of The Function F X Y Z 3x 2 2y 2 Z 2 1 2

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