Find The Length Of The Curve. R(t) = Cos(3t) I Sin(3t) J 3 Ln(cos(t)) K, 0 T /4

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Calculating the length of a parametric curve is a fundamental problem in calculus, with applications spanning physics, engineering, and computer graphics. The given vector function, R(t) = Cos(3t) I + Sin(3t) J + 3 Ln(cos(t)) K, defines a space curve over the interval t ∈ [0, T/4]. In this comprehensive guide, we will walk through the process of finding the exact length of this curve step-by-step, ensuring clarity and depth for learners and professionals alike. We will discuss the mathematical background, derive the formula, compute derivatives, and evaluate the integral to determine the total arc length.

Understanding the Parametric Curve R(t)

Before diving into calculations, it is essential to understand the components of the curve:

- Vector Function Definition:

R(t) = x(t) I + y(t) J + z(t) K

where:

- x(t) = cos(3t)
- y(t) = sin(3t)
- z(t) = 3 ln(cos(t))

- Interval of t:

t ranges from 0 to T/4, where T is the period of the functions involved. Notably, since cos(t) appears inside the natural logarithm, the domain of t must be restricted to ensure cos(t) > 0 (because ln(cos(t)) is only real-valued when cos(t) > 0).

- Domain Considerations:

- cos(t) > 0 when t ∈ (−π/2, π/2). Since t ∈ [0, T/4], T/4 must be ≤ π/2 for the function to be well-defined over the entire interval.

- For simplicity, assume T = π, so that t ∈ [0, π/4], which satisfies the domain restrictions.

Mathematical Background: Arc Length of a Parametric Curve

The length L of a space curve R(t) from t = a to t = b is given by the integral:

$$
L = \int_a^b \left| R'(t) \right| dt
$$

where $\left| R'(t) \right|$ is the magnitude of the derivative of R(t).

- Derivative of R(t):

R'(t) = $\frac{d}{dt} R(t) = \frac{d}{dt} [x(t), y(t), z(t)]$

- Magnitude of R'(t):

$$
| R'(t) | = \sqrt{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 }
$$

The core of the problem is to compute R'(t), find its magnitude, and integrate over the interval [0, T/4].

Step-by-Step Calculation of the Derivative R'(t)

Let's compute each component derivative:

1. Derivative of x(t) = cos(3t)

$$
\frac{dx}{dt} = -3 \sin(3t)
$$

2. Derivative of y(t) = sin(3t)

$$
\frac{dy}{dt} = 3 \cos(3t)
$$

3. Derivative of z(t) = 3 ln(cos(t))

Recall that:

$$
\frac{d}{dt} \ln(\cos(t)) = - \tan(t)
$$

Thus,

$$
\frac{dz}{dt} = 3 \times \left(- \tan(t) \right) = -3 \tan(t)
$$

Calculating the Magnitude | R'(t) |

Now, the magnitude is:

$$
| R'(t) | = \sqrt{ \left(-3 \sin(3t) \right)^2 + \left(3 \cos(3t) \right)^2 + \left(-3 \tan(t) \right)^2 }
$$

Simplify each term:

- $\left(-3 \sin(3t)\right)^2 = 9 \sin^2(3t)$
- $\left(3 \cos(3t)\right)^2 = 9 \cos^2(3t)$
- $\left(-3 \tan(t)\right)^2 = 9 \tan^2(t)$

Putting it all together:

$$
| R'(t) | = \sqrt{ 9 \sin^2(3t) + 9 \cos^2(3t) + 9 \tan^2(t) }
$$

Factor out 9:

$$
| R'(t) | = \sqrt{ 9 \left(\sin^2(3t) + \cos^2(3t) + \tan^2(t) \right) } = 3 \sqrt{ \sin^2(3t) + \cos^2(3t) + \tan^2(t) }
$$

Recall that $\sin^2(3t) + \cos^2(3t) = 1$, so:

$$
| R'(t) | = 3 \sqrt{ 1 + \tan^2(t) }
$$

Using the Pythagorean identity:

$$
1 + \tan^2(t) = \sec^2(t)
$$

\]

Therefore,

$$
| R'(t) | = 3 \sqrt{ \sec^2(t) } = 3 |\sec(t)|
$$

Since $t \in [0, \pi/4]$, where $\cos(t) > 0$, $\sec(t) = 1 / \cos(t) > 0$. So the absolute value can be dropped:

$$
| R'(t) | = \frac{3}{\cos(t)}
$$

Formulating the Arc Length Integral

The total length L from $t = 0$ to $t = T/4$ is:

$$
L = \int_0^{T/4} | R'(t) | dt = \int_0^{T/4} \frac{3}{\cos(t)} dt
$$

Choosing $T = \pi$ (for simplicity), the limits become:

$$
t \in [0, \pi/4]
$$

Thus,

$$
L = 3 \int_0^{\pi/4} \frac{1}{\cos(t)} dt = 3 \int_0^{\pi/4} \sec(t) dt
$$

Evaluating the Integral of $\sec(t)$

The integral of $\sec(t)$ is a standard result:

$$
\int \sec(t) dt = \ln | \sec(t) + \tan(t) | + C
$$

Applying the limits:

$$
L = 3 \left[\ln | \sec(t) + \tan(t) | \right]_0^{\pi/4}
$$

Compute at t = $\pi/4$:

- $\cos(\pi/4) = \frac{\sqrt{2}}{2}$
- $\sec(\pi/4) = \frac{1}{\cos(\pi/4)} = \sqrt{2}$
- $\tan(\pi/4) = 1$

Thus,

$$
\sec(\pi/4) + \tan(\pi/4) = \sqrt{2} + 1
$$

At t = 0:

- $\cos(0) = 1$
- $\sec(0) = 1$
- $\tan(0) = 0$

So,

$$
\sec(0) + \tan(0) = 1 + 0 = 1
$$

Putting it all together:

$$
L = 3 \left[\ln (\sqrt{2} + 1) - \ln(1) \right] = 3 \ln (\sqrt{2} + 1)
$$

Final Expression for the Curve Length

The length of the curve R(t) over the interval [0, π/4] is:

$$
\boxed{
L = 3 \ln (\sqrt{2} + 1)
}
$$

This result provides a precise measure of the length of the specified space curve segment.

Additional Considerations and Applications

Domain Restrictions

- The derivation assumes $t \in [0, \pi/4]$, ensuring $\cos(t) > 0$ for the logarithm to be defined.
- If T differs, adjust the limits accordingly, ensuring the domain remains valid.

Practical Applications

- Physics: Calculating the length of a particle's trajectory in space.
- Engineering: Designing curves with specific lengths for manufacturing or structural purposes.
- Computer Graphics: Rendering smooth curves with accurate length measurements

Frequently Asked Questions

How do you find the length of the curve R(t) = Cos(3t) I Sin(3t) J 3 Ln(cos(t)) K over the interval 0 to π/4?

To find the length, compute the derivative R'(t), then evaluate the integral of the magnitude |R'(t)| from 0 to π/4: Length = $\int_0$^{π/4} |R'(t)| dt.

What is the first step in calculating the arc length of the given vector function?

The first step is to differentiate each component of R(t) with respect to t to find R'(t).

How do you handle the derivative of the component 3 Ln(cos(t)) in R(t)?

Use the chain rule: d/dt [3 ln(cos(t))] = 3 (1 / cos(t)) (-sin(t)) = -3 tan(t).

What is the magnitude |R'(t)| in this context?

The magnitude |R'(t)| is the square root of the sum of the squares of the derivatives of each component: sqrt[(d/dt Cos(3t))^2 + (d/dt Sin(3t))^2 + (d/dt 3 ln(cos(t)))^2].

Are there any special considerations or simplifications when integrating |R'(t)| over [0, π/4]?

Yes, simplifying the derivatives before integration helps, especially noting that certain terms like 3 tan(t) can be integrated more straightforwardly, and recognizing that cos(3t) and sin(3t) derivatives involve sine and cosine functions.

What is the significance of the interval 0 to π/4 in calculating the curve length?

This interval specifies the parameter t over which the length is calculated, capturing the segment of the curve from $t=0$ to $t=\pi/4$, where the behavior of the functions is well-defined and avoids singularities like $\tan(t)$ at $t=\pi/2$.

Additional Resources

Find The Length Of The Curve. R(t) = Cos(3t) I Sin(3t) J 3 Ln(cos(t)) K, 0 ≤ t ≤ π/4

Introduction

Calculating the length of a parametric curve is a fundamental problem in calculus, with applications spanning physics, engineering, computer graphics, and more. When dealing with complex vector functions, especially those involving transcendental functions like logarithms and trigonometric compositions, the task becomes both an academic challenge and an analytical adventure. This article offers a comprehensive, investigative approach to finding the length of the curve defined by the vector function:

$$
\mathbf{R}(t) = \cos(3t)\,\mathbf{i} + \sin(3t)\,\mathbf{j} + 3 \ln(\cos t)\,\mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{4}
$$

We will explore the process step-by-step, including derivative calculations, integrand formulation, potential pitfalls, and interpretations, ensuring a deep understanding of the problem's structure.

Understanding the Problem and Its Significance

The task of determining the length of a parametric curve like $\mathbf{R}(t)$ involves integrating the magnitude of its derivative over the specified interval. This problem is emblematic of more complex calculus exercises where multiple functions are intertwined, such as oscillatory components $\cos(3t)$, $\sin(3t)$, and the logarithmic term $3 \ln(\cos t)$.

Calculating the arc length L of the curve over $t \in [0, \pi/4]$ is given by:

$$
L = \int_{0}^{\pi/4} |\mathbf{R}'(t)|\, dt
$$

where

$$
|\mathbf{R}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}
$$

This formulation hinges on the derivative components, which, given the composite functions involved, require meticulous calculation and simplification.

Component-Wise Derivation of $\mathbf{R}'(t)$

Before tackling the integral, understanding the derivatives of each component is essential.

Derivative of $x(t) = \cos(3t)$

$$
\frac{dx}{dt} = -3 \sin(3t)
$$

This is straightforward, relying on the chain rule with the outer derivative of cosine.

Derivative of $y(t) = \sin(3t)$

$$
\frac{dy}{dt} = 3 \cos(3t)
$$

Again, a direct application of the chain rule.

Derivative of $z(t) = 3 \ln(\cos t)$

This component is more intricate due to the logarithm:

$$
\frac{dz}{dt} = 3 \times \frac{1}{\cos t} \times (-\sin t) = -3 \frac{\sin t}{\cos t} = -3 \tan t
$$

Thus,

\[

$$
\frac{dz}{dt} = -3 \tan t
$$

Constructing the Magnitude of $\mathbf{R}'(t)$

The next step involves computing:

$$
|\mathbf{R}'(t)| = \sqrt{ \left(-3 \sin(3t)\right)^2 + \left(3 \cos(3t)\right)^2 + \left(-3 \tan t\right)^2 }
$$

Simplify each term:

$$
(-3 \sin(3t))^2 = 9 \sin^2(3t)
$$
$$
(3 \cos(3t))^2 = 9 \cos^2(3t)
$$
$$
(-3 \tan t)^2 = 9 \tan^2 t
$$

Therefore,

$$
|\mathbf{R}'(t)| = \sqrt{ 9 \sin^2(3t) + 9 \cos^2(3t) + 9 \tan^2 t }
$$

Factor out 9:

$$
|\mathbf{R}'(t)| = 3 \sqrt{ \sin^2(3t) + \cos^2(3t) + \tan^2 t }
$$

Recognize that:

$$
\sin^2(3t) + \cos^2(3t) = 1
$$

So the expression simplifies to:

$$
|\mathbf{R}'(t)| = 3 \sqrt{ 1 + \tan^2 t }
$$

Recall the fundamental trigonometric identity:

$$
1 + \tan^2 t = \sec^2 t
$$

Thus,

$$
|\mathbf{R}'(t)| = 3 \sqrt{\sec^2 t} = 3 |\sec t|
$$

Since $t \in [0, \pi/4]$, and $\cos t > 0$ in this interval, $\sec t = 1/\cos t > 0$. Therefore:

$$
|\mathbf{R}'(t)| = 3 \sec t
$$

Formulating and Evaluating the Arc Length Integral

The arc length is:

$$
L = \int_{0}^{\pi/4} |\mathbf{R}'(t)|\, dt = \int_{0}^{\pi/4} 3 \sec t\, dt
$$

Pulling out the constant:

$$
L = 3 \int_{0}^{\pi/4} \sec t\, dt
$$

Integral of $\sec t$:

The integral of $\sec t$ is well-known:

$$
\int \sec t\, dt = \ln |\sec t + \tan t| + C
$$

Applying the limits:

$$
L = 3 \left[\ln |\sec t + \tan t| \right]_0^{\pi/4}
$$

Computing the Definite Integral

Evaluate at $t = \pi/4$:

$$
\sec \left(\frac{\pi}{4}\right) = \frac{1}{\cos(\pi/4)} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}
$$
$$
\tan \left(\frac{\pi}{4}\right) = 1
$$

Thus,

$$
\sec \left(\frac{\pi}{4}\right) + \tan \left(\frac{\pi}{4}\right) = \sqrt{2} + 1
$$

At $t = 0$:

$$
\sec 0 = 1, \quad \tan 0 = 0
$$

So,

$$
\sec 0 + \tan 0 = 1 + 0 = 1
$$

Putting it all together:

$$
L = 3 \left[\ln (\sqrt{2} + 1) - \ln 1 \right] = 3 \ln (\sqrt{2} + 1)
$$

Since $\ln 1 = 0$, the final length of the curve is:

$$
\boxed{
\textbf{Length } L = 3 \ln (\sqrt{2} + 1)
}
$$

Discussion and Implications

This result manifests the elegance of calculus and the power of recognizing identities. The initial complex expression simplifies neatly owing to fundamental trigonometric identities, transforming what appears to be a complicated integral into a straightforward calculation.

The key steps involved:

- Deriving the derivatives component-wise.
- Recognizing the Pythagorean identity to simplify the magnitude.
- Applying the integral of $\sec t$ with proper limits.
- Confirming the positivity of $\sec t$ over the interval to avoid absolute value complications.

This process underscores the importance of understanding basic identities and properties, which can significantly streamline calculations involving complicated functions.

Potential Extensions and Applications

While this calculation provides an exact expression for the length, several avenues extend from here:

- Numerical Approximation: For more complex functions where integral forms cannot be expressed explicitly, numerical methods like Simpson's rule or Gaussian quadrature are invaluable.
- Graphical Analysis: Visualizing the curve in three dimensions can offer geometric insights into its shape and length.
- Parameter Variations: Adjusting the bounds or the functions involved can help model various physical phenomena, such as the length of a wire shaped according to a specific parametrization.
- Higher-Dimensional Analogues: Extending these techniques to manifolds and higher-dimensional curves.

Conclusion

The process of finding the length of the curve $\mathbf{R}(t) = \cos(3t) \mathbf{i} + \sin(3t) \mathbf{j} + 3 \ln(\cos t) \mathbf{k}$ over $t \in [0, \pi/4]$ exemplifies the beautiful interplay between calculus, algebra, and trigonometry. Through careful derivative calculation, recognition of

[Find The Length Of The Curve R T Cos 3t I Sin 3t J 3 Ln Cos T K 0 T 4]

Find The Length Of The Curve R T Cos 3t I Sin 3t J 3 Ln Cos T K 0 T 4

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