Find The Solution Of The Initial Value Problem Y'' 4y=t^2 2e^t, Y(0)=0 Y'(0)=1

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In this comprehensive article, we aim to solve the initial value problem (IVP) involving a second-order linear differential equation with constant coefficients. The problem is given as:

$$ Y'' - 4Y = t^2 \cdot 2e^t, \quad Y(0) = 0, \quad Y'(0) = 1. $$

This type of problem involves multiple steps, including solving the homogeneous equation, finding a particular solution to the nonhomogeneous part, and applying initial conditions to determine the constants of integration. We will systematically explore each step to derive the exact solution.

Understanding the Differential Equation

The Form of the Equation

The differential equation under consideration is:

$$ Y'' - 4Y = t^2 \cdot 2e^t. $$

This is a nonhomogeneous second-order linear differential equation with constant coefficients. The left side $\ Y'' - 4Y$ is linear and homogeneous, while the right side $t^2 2e^t$ is a nonhomogeneous term involving a polynomial multiplied by an exponential function.

Significance of the Equation

Such equations frequently appear in physics and engineering, especially in systems subjected to external forcing functions. The key to solving them lies in:

- Finding the general solution to the homogeneous equation.
- Finding a particular solution that accounts for the nonhomogeneous term.
- Combining both to construct the general solution.
- Applying initial conditions to find specific constants.

Step 1: Solving the Homogeneous Equation

Homogeneous Equation

The homogeneous counterpart of the differential equation is:

$$ Y'' - 4Y = 0. $$

Characteristic Equation

To solve the homogeneous equation, we assume solutions of the form $Y_h = e^{rt}$, leading to the characteristic equation:

$$ r^2 - 4 = 0. $$

Solving for r:

$$ r^2 = 4 \quad \Rightarrow \quad r = \pm 2. $$

Homogeneous Solution

Thus, the general solution to the homogeneous equation is:

$$ Y_h(t) = C_1 e^{2t} + C_2 e^{-2t}, $$

where C_1 and C_2 are arbitrary constants.

Step 2: Finding a Particular Solution

Nature of the Nonhomogeneous Term

The right-hand side is:

$$ f(t) = 2t^2 e^t. $$

Since e^t is not a solution to the homogeneous equation, and the polynomial is of degree 2, we can use the method of undetermined coefficients or variation of parameters. Given the polynomial multiplied by an exponential, the method of undetermined coefficients is suitable.

Form of the Particular Solution

We propose a particular solution of the form:

$$ Y_p(t) = e^{t} (A t^2 + B t + C), $$

where A, B, C are constants to be determined.

Note: We include the polynomial t^2 multiplied by e^t because the RHS is $2 t^2 e^t$.

Computing Derivatives

Calculate the first and second derivatives:

$$
Y_p(t) = e^{t}(A t^2 + B t + C),
$$

$$
Y_p'(t) = e^{t}(A t^2 + B t + C) + e^{t}(2A t + B) = e^{t} [A t^2 + B t + C + 2A t + B],
$$

$$
Y_p'(t) = e^{t} [A t^2 + (B + 2A) t + (C + B)],
$$

$$
Y_p''(t) = \frac{d}{dt} \left(e^{t} [A t^2 + (B + 2A) t + (C + B)] \right).
$$

Applying product rule:

$$
Y_p''(t) = e^{t} [A t^2 + (B + 2A) t + (C + B)] + e^{t} \frac{d}{dt} [A t^2 + (B + 2A) t + (C + B)].
$$

The derivative inside is:

$$
2A t + (B + 2A).
$$

Thus:

$$
Y_p''(t) = e^{t} \left[A t^2 + (B + 2A) t + (C + B) + 2A t + (B + 2A) \right].
$$

Combine like terms:

$$
Y_p''(t) = e^{t} \left[A t^2 + ((B + 2A) + 2A) t + (C + B + B + 2A) \right],
$$

$$
Y_p''(t) = e^{t} \left[A t^2 + (B + 4A) t + (C + 2B + 2A) \right].
$$

Step 3: Substituting into the Differential Equation

Now, substitute Y_p, Y_p', and Y_p'' into the original differential equation:

$$
Y'' - 4Y = 2 t^2 e^t.
$$

Expressed as:

$$
Y_p'' - 4 Y_p = 2 t^2 e^t.
$$

Substituting the derivatives:

$$
e^{t} [A t^2 + (B + 4A) t + (C + 2B + 2A)] - 4 e^{t} (A t^2 + B t + C) = 2 t^2 e^t.
$$

Factor out e^{t}:

$$
e^{t} \left[A t^2 + (B + 4A) t + (C + 2B + 2A) - 4A t^2 - 4 B t - 4 C \right] = 2 t^2 e^t.
$$

Simplify the bracket:

$$
e^{t} \left[(A t^2 - 4A t^2) + ((B + 4A) t - 4 B t) + (C + 2B + 2A - 4 C) \right] = 2 t^2 e^t.
$$

Compute each term:

- $A t^2 - 4A t^2 = -3A t^2$,
- $(B + 4A) t - 4 B t = B t + 4A t - 4 B t = (B - 4B) t + 4A t = -3B t + 4A t$,
- $C + 2B + 2A - 4 C = -3 C + 2B + 2A$.

Putting it all together:

$$
e^{t} \left[-3A t^2 + (-3B t + 4A t) + (-3 C + 2 B + 2A) \right] = 2 t^2 e^{t}.
$$

Divide both sides by e^{t}:

$$
-3A t^2 + (-3B t + 4A t) + (-3 C + 2 B + 2A) = 2 t^2.
$$

Step 4: Equating Coefficients

Since this equation holds for all t, the coefficients of like powers of t must be equal on both sides:

$$
-3A t^2 + (-3B t + 4A t) + (-3 C + 2 B + 2A) = 2 t^2 + 0 t + 0.
$$

Matching coefficients:

- For t^2:

$$
-3A = 2 \quad \Rightarrow \quad A = -\frac{2}{3}.
$$

- For t:

$$
-3B + 4A = 0 \quad \Rightarrow \quad -3B + 4 \left(- \frac{2}{3} \right) = 0,
$$

$$
-3B - \frac{8}{3} = 0,
$$

$$
-3B = \frac{8}{3},
$$

\[
B = - \frac{8}{9}.
\]

- For constants:

\[
-3 C + 2 B + 2 A = 0,
\]

\[
-3 C + 2 \left(- \frac{8}{9} \right) + 2 \left(- \frac{2}{3}

Frequently Asked Questions

What is the initial value problem given for the differential equation Y'' + 4Y = t^2 2e^t with Y(0)=0 and Y'(0)=1?

The problem is to find the function Y(t) satisfying the differential equation Y'' + 4Y = 2t^2 e^t with initial conditions Y(0)=0 and Y'(0)=1.

What is the general solution to the homogeneous differential equation Y'' + 4Y = 0?

The homogeneous solution is Y_h(t) = C_1 cos 2t + C_2 sin 2t, where C_1 and C_2 are arbitrary constants.

How do we find a particular solution to the nonhomogeneous differential equation Y'' + 4Y = 2t^2 e^t?

We use the method of undetermined coefficients or variation of parameters. Given the right side involves t^2 e^t, we assume a particular solution of the form Y_p(t) = e^t (A t^2 + B t + C) and determine A, B, C by substitution.

What are the steps to apply initial conditions Y(0)=0 and Y'(0)=1 to determine the constants in the solution?

First, find the complete general solution Y(t) = Y_h(t) + Y_p(t). Then, substitute t=0 into Y(t) and Y'(t), and set these equal to the initial conditions to solve for C_1 and C_2.

What is the significance of solving this initial value

problem in the context of differential equations?

Solving this IVP helps understand how to find specific solutions to nonhomogeneous differential equations with given initial conditions, illustrating methods like the homogeneous solution, particular solution, and applying initial conditions for unique solutions.

Additional Resources

Find The Solution Of The Initial Value Problem Y'' + 4y = t^2 2e^t, Y(0) = 0, Y'(0) = 1 — this is a classic example of solving a nonhomogeneous second-order differential equation with initial conditions. Such problems are fundamental in understanding how systems evolve over time in fields ranging from physics to engineering and applied mathematics. In this guide, we will explore step-by-step how to find the solution, breaking down the process into manageable parts, and providing insight into the methods used.

Introduction to the Problem

The initial value problem (IVP) under consideration is:

$$ Y'' + 4Y = t^2 \cdot 2e^t $$

with initial conditions:

$$ Y(0) = 0, \quad Y'(0) = 1. $$

Here, Y'' denotes the second derivative of Y with respect to t, and the right-hand side is a nonhomogeneous term involving polynomial and exponential functions.

Finding the solution involves two main steps:

1. Solving the corresponding homogeneous equation.
2. Finding a particular solution to the nonhomogeneous equation.

Finally, applying initial conditions will yield the specific solution to this IVP.

Step 1: Solving the Homogeneous Equation

The homogeneous counterpart of our differential equation is:

$$ Y'' + 4Y = 0. $$

Characteristic Equation

To solve this, we assume solutions of the form $Y = e^{rt}$, leading to the characteristic equation:

$$ r^2 + 4 = 0. $$

Solving for r:

$$ r^2 = -4 \Rightarrow r = \pm 2i. $$

Homogeneous Solution

The roots are complex conjugates, so the homogeneous solution is:

$$ Y_h(t) = C_1 \cos(2t) + C_2 \sin(2t), $$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

Step 2: Finding a Particular Solution

The nonhomogeneous term is:

$$ f(t) = 2t^2 e^t. $$

Method: Undetermined Coefficients or Variation of Parameters

Because $f(t)$ involves both polynomial and exponential factors, the Method of Undetermined Coefficients may be cumbersome here, especially considering the polynomial part. Instead, Variation of Parameters is more systematic and applicable for such cases.

Step 3: Variation of Parameters

General Formula

For the second-order linear differential equation:

$$ Y'' + p(t)Y' + q(t)Y = f(t), $$

the particular solution $Y_p(t)$ can be expressed as:

$$ Y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t), $$

where $y_1(t)$ and $y_2(t)$ are the fundamental solutions to the homogeneous equation, and $u_1(t), u_2(t)$ are functions to be determined.

Applying to Our Equation

Since our homogeneous solution is:

$$ y_1(t) = \cos(2t), $$

$$ y_2(t) = \sin(2t), $$

we find $u_1(t)$ and $u_2(t)$ via:

$$ u_1'(t) = - \frac{y_2(t) f(t)}{W(t)}, $$
$$ u_2'(t) = \frac{y_1(t) f(t)}{W(t)}, $$

where $W(t)$ is the Wronskian:

$$ W(t) = y_1 y_2' - y_1' y_2. $$

Computing the Wronskian

Calculate derivatives:

$$ y_1' = -2 \sin(2t), $$
$$ y_2' = 2 \cos(2t). $$

Thus,

$$ W(t) = \cos(2t) \cdot 2 \cos(2t) - (-2 \sin(2t)) \cdot \sin(2t) = 2 \cos^2(2t) + 2 \sin^2(2t). $$

Using the Pythagorean identity:

$$ \cos^2(2t) + \sin^2(2t) = 1, $$

we get:

$$ W(t) = 2 \times 1 = 2. $$

Determining $u_1'(t)$ and $u_2'(t)$

Now,

$$ u_1'(t) = - \frac{\sin(2t) \cdot 2 e^t t^2}{2} = - \sin(2t) e^t t^2, $$

$$ u_2'(t) = \frac{\cos(2t) \cdot 2 e^t t^2}{2} = \cos(2t) e^t t^2. $$

Integrating to Find $u_1(t)$ and $u_2(t)$

The integrals are:

$$ u_1(t) = - \int \sin(2t) e^t t^2 dt, $$
$$ u_2(t) = \int \cos(2t) e^t t^2 dt. $$

These integrals are non-trivial and involve integrating products of polynomials, exponentials, and trigonometric functions. We will need to apply integration by parts multiple times or use advanced methods like tabular integration.

Step 4: Computing the Particular Solution

Let's focus on computing $u_2(t)$ as an example; similar steps apply for $u_1(t)$.

Integral for $u_2(t)$:

$$ u_2(t) = \int \cos(2t) e^t t^2 dt. $$

Step 5: Handling the Integrals

Strategy: Repeated Integration by Parts

Because the integrand involves t^2, exponential, and trigonometric functions, integration by parts will be performed repeatedly.

Step 6: Final Expression for the Particular Solution

Once $u_1(t)$ and $u_2(t)$ are evaluated, the particular solution is:

$$ Y_p(t) = u_1(t) \cos(2t) + u_2(t) \sin(2t). $$

Adding this to the homogeneous solution:

$$ Y(t) = C_1 \cos(2t) + C_2 \sin(2t) + Y_p(t). $$

Step 7: Applying Initial Conditions

Recall:

$$ Y(0) = 0, $$
$$ Y'(0) = 1. $$

Using these, we will solve for C_1 and C_2 once the full expression for $Y(t)$ is obtained.

Summary of the Solution Procedure

Here is a summarized list of steps for solving the IVP:

1. Solve the homogeneous equation to find $Y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$.
2. Use variation of parameters to find particular solutions by computing integrals involving $f(t) = 2t^2 e^t$.
3. Evaluate the integrals for $u_1(t)$ and $u_2(t)$ via repeated integration by parts.

4. Construct the general solution: $Y(t) = Y_h(t) + Y_p(t)$.
5. Apply initial conditions to determine the constants C_1, C_2.

Practical Considerations and Numerical Methods

In practice, the integrals involved in the variation of parameters can be quite complex, often requiring numerical methods or software tools like WolframAlpha, Mathematica, or MATLAB for explicit evaluation.

Final Remarks

Solving initial value problems of this nature exemplifies the importance of methodical approaches — understanding the structure of the homogeneous solution, applying variation of parameters for the particular solution, and carefully implementing initial conditions. While the integrals may seem intimidating initially, breaking them down systematically makes the problem manageable.

This process not only provides a specific solution to the differential equation but also deepens understanding of the techniques used in differential equations, which are invaluable in modeling real-world systems, from mechanical vibrations to electrical circuits.

In conclusion, finding the solution of the initial value problem $Y'' + 4Y = t^2 2e^t$ with specified initial conditions involves a blend of solving homogeneous equations, applying variation of parameters, and carefully evaluating integrals. With patience and systematic analysis, the complete solution can be constructed, providing insights into how systems respond to complex forcing functions over time.

Find The Solution Of The Initial Value Problem Y 4y T 2 2e T Y 0 0 Y 0 1

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