Find The Solution To The Initial-value Problem 3xy' - 5y = 7x^3; Y(1) = C. ; Y =

Find The Solution To The Initial-value Problem 3xy' - 5y = 7x^3; Y(1) = C. ; Y = is a common problem encountered in differential equations, particularly in the study of first-order linear differential equations. In this article, we will explore step-by-step how to solve this initial-value problem (IVP), understand the underlying methods, and interpret the solution. Whether you're a student preparing for exams or a professional applying differential equations in modeling, understanding this problem will enhance your problem-solving toolkit.

Understanding the Initial-value Problem

Before diving into the solution, it's essential to understand what the problem entails. The differential equation provided is:

$$3xy' - 5y = 7x^3$$

with the initial condition:

$$Y(1) = C$$

and the goal is to find the explicit form of the function \(Y = y(x)\) satisfying both the differential equation and the initial condition.

Rearranging and Recognizing the Type of Differential Equation

Standard Form of a First-Order Linear Differential Equation

A first-order linear differential equation has the general form:

$$y' + P(x) y = Q(x)$$

To analyze our problem, we need to rewrite the given differential equation into this standard form.

Given:

$$3xy' - 5y = 7x^3$$

Divide both sides by 3x (assuming $x \neq 0$):

$$y' - \frac{5}{3x} y = \frac{7x^2}{3}$$

Now, the equation is in the form:

$$y' + P(x) y = Q(x)$$

where:

- $P(x) = -\frac{5}{3x}$
- $Q(x) = \frac{7x^2}{3}$

Type Identification

This is a linear first-order differential equation that can be solved using an integrating factor method.

Methodology for Solving the Differential Equation

Step 1: Find the Integrating Factor (IF)

The integrating factor is given by:

$$\mu(x) = e^{\int P(x) dx}$$

Compute:

$$
\int P(x) dx = \int -\frac{5}{3x} dx = -\frac{5}{3} \int \frac{1}{x} dx = -\frac{5}{3} \ln|x|
$$

Thus:

$$
\mu(x) = e^{-\frac{5}{3} \ln|x|} = |x|^{-\frac{5}{3}}
$$

Since we're considering $x > 0$ (from initial condition $x=1$), we can write:

$$
\mu(x) = x^{-\frac{5}{3}}

\]

Step 2: Multiply the entire differential equation by the integrating factor

Multiplying:

\[
x^{-\frac{5}{3}} y' - \frac{5}{3x} x^{-\frac{5}{3}} y = \frac{7x^2}{3} x^{-\frac{5}{3}}
\]

which simplifies to:

\[
\frac{d}{dx} \left(x^{-\frac{5}{3}} y \right) = \frac{7x^2}{3} x^{-\frac{5}{3}}
\]

because the left side is the derivative of \(\mu(x) y \).

Step 3: Integrate both sides to find \(y(x) \)

Integrate:

\[
x^{-\frac{5}{3}} y = \int \frac{7x^2}{3} x^{-\frac{5}{3}} dx + K
\]

Express the integrand:

\[
\frac{7}{3} x^{2} \cdot x^{-\frac{5}{3}} = \frac{7}{3} x^{2 - \frac{5}{3}} = \frac{7}{3} x^{\frac{6}{3} - \frac{5}{3}} = \frac{7}{3} x^{\frac{1}{3}}
\]

Now, the integral becomes:

\[
\int \frac{7}{3} x^{\frac{1}{3}} dx = \frac{7}{3} \int x^{\frac{1}{3}} dx
\]

Recall:

\[
\int x^{n} dx = \frac{x^{n+1}}{n+1} + C
\]

So:

$$
\frac{7}{3} \cdot \frac{x^{\frac{1}{3} + 1}}{\frac{1}{3} + 1} = \frac{7}{3} \cdot \frac{x^{\frac{4}{3}}}{\frac{4}{3}} = \frac{7}{3} \times \frac{3}{4} x^{\frac{4}{3}} = \frac{7}{4} x^{\frac{4}{3}}
$$

Therefore,

$$
x^{-\frac{5}{3}} y = \frac{7}{4} x^{\frac{4}{3}} + K
$$

where K is the constant of integration.

Expressing the General Solution

Multiply both sides by $x^{\frac{5}{3}}$:

$$
y(x) = x^{\frac{5}{3}} \left(\frac{7}{4} x^{\frac{4}{3}} + K \right) = \frac{7}{4} x^{\frac{5}{3} + \frac{4}{3}} + K x^{\frac{5}{3}}
$$

Simplify exponents:

$$
\frac{5}{3} + \frac{4}{3} = \frac{9}{3} = 3
$$

Thus,

$$
\boxed{
Y(x) = y(x) = \frac{7}{4} x^{3} + C x^{\frac{5}{3}}
}
$$

where C is an arbitrary constant (replacing K).

Applying the Initial Condition

Given:

\[

$$
Y(1) = C
$$

Substitute $x=1$:

$$
Y(1) = \frac{7}{4} \times 1^{3} + C \times 1^{\frac{5}{3}} = \frac{7}{4} + C
$$

Since the initial condition specifies $Y(1) = C$, we have:

$$
C = \frac{7}{4} + C
$$

This implies:

$$
C = \frac{7}{4} + C
$$

which leads to:

$$
0 = \frac{7}{4}
$$

This indicates a contradiction unless the notation is clarified. To avoid confusion, let's denote the arbitrary constant in the general solution as k, and the constant in the initial condition as C:

- General solution:

$$
Y(x) = \frac{7}{4} x^{3} + k x^{\frac{5}{3}}
$$

- Initial condition:

$$
Y(1) = C
$$

Substitute:

$$
C = \frac{7}{4} \times 1^{3} + k \times 1^{\frac{5}{3}} = \frac{7}{4} + k
$$

Solve for k:

$$
k = C - \frac{7}{4}
$$

Final solution incorporating initial condition:

$$
\boxed{
Y(x) = \frac{7}{4} x^{3} + \left(C - \frac{7}{4}\right) x^{\frac{5}{3}}
}
$$

This represents the explicit solution to the initial-value problem.

Summary of the Solution Process

- Rearranged the differential equation into standard linear form.
- Calculated the integrating factor $\mu(x) = x^{-\frac{5}{3}}$.
- Multiplied through by the integrating factor to facilitate integration.
- Integrated both sides to find the general solution.
- Applied the initial condition to determine the specific solution constant.

Additional Insights and Applications

Why Is This Solution Important?

Understanding how to solve this IVP provides foundational skills applicable in various fields, including physics, engineering, and economics. Differential equations describe systems where change depends on current states, such as population dynamics, heat transfer, and electrical circuits.

Potential Variations and Generalizations

- Different initial conditions: Altering the value of $Y(1)$ changes the particular solution

Frequently Asked Questions

What is the initial-value problem given in the equation 3xy - 5y = 7x^3 with Y(1) = C?

The initial-value problem is to find the function Y(x) satisfying 3xy - 5y = 7x^3 with the condition Y(1) = C.

How do you rewrite the differential equation 3xy - 5y = 7x^3 in a standard form?

First, express it as a differential equation in terms of Y and its derivative. Since the given form is algebraic, we need to identify the relationship between Y and x, possibly rewriting it in terms of derivatives if applicable. However, as presented, it appears to be an algebraic relation involving y, which suggests clarification is needed to determine whether it's a differential equation or an algebraic equation.

Is the given equation 3xy - 5y = 7x^3 a differential equation or an algebraic equation?

The equation 3xy - 5y = 7x^3 appears to be an algebraic relation involving y and x, not a differential equation. If the problem intends for Y to be a function satisfying a differential equation, additional information such as derivatives is needed.

Assuming the equation involves derivatives, how would you solve a differential equation similar to 3xy - 5y = 7x^3?

Typically, for a differential equation, you would separate variables or use integrating factors. For example, if the equation were in the form dy/dx = f(x, y), you'd attempt to rewrite and integrate accordingly. In this case, more context is needed to determine the exact differential form.

How do you find the particular solution given the initial condition Y(1) = C?

Once the general solution of the differential equation is found, substitute x = 1 and Y = C into it to solve for the constant of integration, thus obtaining the particular solution.

What are common methods to solve initial-value problems involving algebraic equations like 3xy - 5y = 7x^3?

If the equation is algebraic, methods include solving for y explicitly or using substitution if it's part of a larger differential equation. If it’s a differential equation, techniques like separation of variables, integrating factors, or substitution are used.

Can you clarify whether the original problem is a differential equation or an algebraic relation?

Based on the given form 3xy - 5y = 7x^3 with Y(1) = C, it appears to be an algebraic relation rather than a differential equation. If the problem is to find Y(x), additional information about derivatives is necessary to proceed with a differential equation approach.

Additional Resources

Find The Solution To The Initial-value Problem 3xy' - 5y = 7x^3; Y(1) = C. ; Y =

Understanding how to solve differential equations is fundamental in applied mathematics, physics, engineering, and numerous scientific disciplines. Today, we delve into a specific initial-value problem (IVP): solving the differential equation 3xy' - 5y = 7x^3 with the initial condition Y(1) = C. This problem is a classic example of a first-order linear differential equation, which, despite its apparent complexity, can be systematically tackled using established methods. The goal is to find a function Y(x) that satisfies the differential equation and the initial condition, providing a complete solution that can be used to model various real-world phenomena.

Understanding the Problem: Differential Equation and Initial Condition

The problem is posed as follows:

- Differential Equation: $3x y' - 5 y = 7x^3$
- Initial Condition: $y(1) = C$

Here, y' denotes the derivative of y with respect to x. The task is to find a function $y(x)$, expressed explicitly or implicitly, that satisfies both the differential equation and the initial condition.

This type of differential equation belongs to the class of first-order linear equations, which are characterized by the general form:

$$
y' + P(x) y = Q(x)
$$

Our initial goal is to manipulate the given equation into this standard form to facilitate solution methods.

Rewriting the Equation: Standard Form and Recognizing It

Let's start by rewriting the original differential equation:

$$
3x y' - 5 y = 7x^3
$$

Divide through by $3x$ (assuming $x \neq 0$):

$$
y' - \frac{5}{3x} y = \frac{7x^3}{3x} = \frac{7}{3} x^2
$$

Now, the equation is in the standard linear form:

$$
y' + P(x) y = Q(x)
$$

where:

- $P(x) = - \frac{5}{3x}$
- $Q(x) = \frac{7}{3} x^2$

This form is essential because it allows us to apply the integrating factor method, a powerful technique for solving linear differential equations.

Applying the Integrating Factor Method

The integrating factor (IF) is given by:

$$
\mu(x) = e^{\int P(x) dx}
$$

Calculating the integral:

$$
\int P(x) dx = \int -\frac{5}{3x} dx = -\frac{5}{3} \int \frac{1}{x} dx = - \frac{5}{3} \ln|x|
$$

Thus,

$$
\mu(x) = e^{-\frac{5}{3} \ln|x|} = |x|^{-\frac{5}{3}}
$$

Since we're generally considering $x > 0$ (to avoid complications with the absolute value), we simplify:

$$
\mu(x) = x^{-\frac{5}{3}}
$$

The integrating factor transforms the original differential equation into an exact derivative:

$$
\frac{d}{dx} [\mu(x) y] = \mu(x) Q(x)
$$

which allows us to integrate both sides directly.

Solving the Equation: Step-by-Step

1. Multiply through by the integrating factor:

$$
x^{-\frac{5}{3}} y' - \frac{5}{3} x^{-\frac{5}{3} - 1} y = \frac{7}{3} x^2 \times x^{-\frac{5}{3}}
$$

But better to proceed as per the standard method:

$$
\frac{d}{dx} [x^{-\frac{5}{3}} y] = x^{-\frac{5}{3}} \times \frac{7}{3} x^2
$$

2. Express the right side:

$$
x^{-\frac{5}{3}} \times \frac{7}{3} x^{2} = \frac{7}{3} x^{2 - \frac{5}{3}} = \frac{7}{3} x^{\frac{6}{3} - \frac{5}{3}} = \frac{7}{3} x^{\frac{1}{3}}
$$

3. Integrate both sides:

$$
\int \frac{d}{dx} [x^{-\frac{5}{3}} y] dx = \int \frac{7}{3} x^{\frac{1}{3}} dx
$$

$$
x^{-\frac{5}{3}} y = \frac{7}{3} \int x^{\frac{1}{3}} dx + K
$$

where K is the constant of integration.

4. Compute the integral:

\[

\int x^{1/3} dx = \frac{x^{1/3 + 1}}{1/3 + 1} = \frac{x^{4/3}}{4/3} = \frac{3}{4} x^{4/3}
\]

5. Write the general solution:

\[
x^{-\frac{5}{3}} y = \frac{7}{3} \times \frac{3}{4} x^{4/3} + K = \frac{7}{4} x^{4/3} + K
\]

6. Solve for \(y \):

\[
y(x) = x^{\frac{5}{3}} \left(\frac{7}{4} x^{4/3} + K \right)
\]

Since:

\[
x^{\frac{5}{3}} \times x^{4/3} = x^{\frac{5}{3} + \frac{4}{3}} = x^{3}
\]

the solution simplifies to:

\[
\boxed{
y(x) = \frac{7}{4} x^{3} + K x^{\frac{5}{3}}
}
\]

Applying the Initial Condition \(y(1) = C \)

To determine the constant \(K \), substitute \(x = 1 \):

\[
y(1) = \frac{7}{4} \times 1^{3} + K \times 1^{5/3} = \frac{7}{4} + K
\]

Given \(y(1) = C \), we have:

\[
C = \frac{7}{4} + K \Rightarrow K = C - \frac{7}{4}
\]

Final explicit solution:

\[
\boxed{

Y(x) = \frac{7}{4} x^3 + \left(C - \frac{7}{4} \right) x^{\frac{5}{3}}
}
\]

This formula encapsulates the entire family of solutions parameterized by the initial value \(C \). It demonstrates how the solution depends linearly on \(C \) and provides a complete description of the behavior of \(y \) with respect to \(x \).

Significance and Applications of the Solution

The explicit solution obtained is more than an abstract mathematical exercise; it has practical implications in numerous fields. For example:

- Physics: Modeling heat transfer or fluid flow where variables depend on position and initial conditions.
- Engineering: Designing control systems where initial states influence future behavior.
- Economics: Modeling growth processes with initial capital or resource levels.

Understanding how to manipulate and solve such differential equations empowers professionals to predict system behavior, optimize processes, and develop models that reflect complex real-world dynamics.

Summary and Key Takeaways

- The initial differential equation was transformed into a standard linear form, enabling systematic solution.
- The integrating factor method was employed to find the general solution.
- The explicit solution depends on an arbitrary constant \(C \), determined by the initial condition.
- The solution's structure reveals how initial conditions influence the overall behavior of the system described by the differential equation.

By mastering this approach, students and professionals alike can confidently tackle similar differential equations, strengthening their analytical toolkit for scientific and engineering challenges.

Final Remarks

Differential equations like \(3x y' - 5 y = 7x^3 \) are cornerstones of mathematical modeling. The process of solving them—rearranging, applying integrating factors, integrating, and applying initial conditions—is both systematic and elegant. As you continue exploring the realm of differential equations, remember that each equation is a gateway to understanding the dynamic systems that shape our world.

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