Solve The Given Differential Equation By Variation Of Parameters. Xy'' 8y' = X8

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Solving differential equations is a fundamental skill in advanced mathematics, physics, engineering, and many applied sciences. Among the various methods available, the variation of parameters stands out as a versatile technique for solving nonhomogeneous linear differential equations, especially when other methods like undetermined coefficients are not suitable. This article provides a comprehensive guide to solving the differential equation Xy'' + 8y' = X8 using the variation of parameters method, including detailed steps, explanations, and practical tips to help you master this technique.

Understanding the Differential Equation

Before diving into the solution process, it's essential to analyze the given differential equation:

$$
X y'' + 8 y' = X8
$$

At first glance, this is a linear second-order differential equation with variable coefficients. To effectively apply the variation of parameters, we need to rewrite it in a standard form and understand its structure.

Key Components of the Equation:

- Dependent variable: $y = y(x)$
- Derivatives: $y' = \frac{dy}{dx}$, $y'' = \frac{d^2 y}{dx^2}$
- Coefficients: X (which appears as a factor in the derivatives), and the constant 8

Standard Form of a Second-Order Linear Differential Equation:

$$
y'' + P(x) y' + Q(x) y = R(x)
$$

Transforming the given equation into this form involves dividing through by X (assuming $X \neq 0$):

$$
y'' + \frac{8}{X} y' = 8
$$

Now, the differential equation is expressed as:

$$
\boxed{
y'' + \frac{8}{x} y' = 8
}
$$

This standard form allows us to identify:

- $P(x) = \frac{8}{x}$
- $Q(x) = 0$
- $R(x) = 8$

Step-by-Step Solution Using Variation of Parameters

The variation of parameters method involves two key steps:

1. Find the general solution to the associated homogeneous equation.
2. Find a particular solution to the nonhomogeneous equation using the variation of parameters.

Let's walk through each step in detail.

Step 1: Solve the Homogeneous Equation

The homogeneous counterpart of the differential equation is:

$$
y'' + \frac{8}{x} y' = 0
$$

This is a linear, second-order homogeneous differential equation with variable coefficients.

Solving the Homogeneous Equation:

We can note that this equation is reducible to a first-order equation by substitution. Let:

$$
z = y'
$$

which gives:

$$
z' + \frac{8}{x} z = 0
$$

This is a first-order linear differential equation in z:

$$
z' + \frac{8}{x} z = 0
$$

Solution process:

- Identify integrating factor:

$$
\mu(x) = \exp \left(\int \frac{8}{x} dx \right) = \exp(8 \ln |x|) = |x|^8
$$

- Multiply through by the integrating factor:

$$
|x|^8 z' + 8 |x|^7 z = 0
$$

which simplifies to:

$$
\frac{d}{dx} (|x|^8 z) = 0
$$

- Integrate:

$$
|x|^8 z = C_1 \quad \Rightarrow \quad z = \frac{C_1}{|x|^8}
$$

Recall that $z = y'$, so:

$$
y' = \frac{C_1}{x^8}
$$

- Integrate to find y:

$$
y = \int y' dx = \int \frac{C_1}{x^8} dx = C_1 \int x^{-8} dx
$$

$$
\int x^{-8} dx = \frac{x^{-7}}{-7} + C_2
$$

Therefore, the general solution to the homogeneous equation is:

$$
\boxed{
y_h = C_1 \left(-\frac{1}{7} x^{-7} \right) + C_2
}
$$

which simplifies to:

$$
\boxed{
y_h = -\frac{C_1}{7} x^{-7} + C_2
}
$$

Step 2: Find a Particular Solution Using Variation of Parameters

The particular solution y_p can be expressed as:

$$
y_p = u_1(x) y_1(x) + u_2(x) y_2(x)
$$

where:

- $y_1(x) = x^{-7}$ (from the homogeneous solution)
- $y_2(x) = 1$

The functions $u_1(x)$ and $u_2(x)$ are to be determined such that:

$$
\begin{cases}
u_1' y_1 + u_2' y_2 = 0 \\
u_1' y_1' + u_2' y_2' = R(x)
\end{cases}
$$

with $R(x) = 8$.

Step 2.1: Compute Wronskian W

The Wronskian W of the fundamental solutions y_1 and y_2:

$$
W = y_1 y_2' - y_1' y_2
$$

$$

Calculate derivatives:

- $y_1 = x^{-7}$, so

$$
y_1' = -7 x^{-8}
$$

- $y_2 = 1$, so

$$
y_2' = 0
$$

Now,

$$
W = x^{-7} \cdot 0 - (-7 x^{-8}) \cdot 1 = 7 x^{-8}
$$

Step 2.2: Find u_1' and u_2'

The formulas for u_1' and u_2':

$$
u_1' = - \frac{y_2 R(x)}{W} = - \frac{1 \times 8}{7 x^{-8}} = - \frac{8}{7} x^{8}
$$

$$
u_2' = \frac{y_1 R(x)}{W} = \frac{x^{-7} \times 8}{7 x^{-8}} = \frac{8 x^{-7}}{7 x^{-8}} = \frac{8}{7} x
$$

Step 2.3: Integrate to find u_1 and u_2

- For u_1:

$$
u_1 = - \frac{8}{7} \int x^{8} dx = - \frac{8}{7} \times \frac{x^{9}}{9} + C_3 = - \frac{8}{63} x^{9} + C_3
$$

- For u_2:

$$

$$
u_2 = \frac{8}{7} \int x dx = \frac{8}{7} \times \frac{x^{2}}{2} + C_4 = \frac{4}{7} x^{2} + C_4
$$

Since the constants C_3 and C_4 are absorbed into the homogeneous solution, we focus on the particular parts.

Constructing the Particular Solution

The particular solution:

$$
y_p = u_1 y_1 + u_2 y_2
$$

becomes:

$$
\boxed{
y_p = \left(- \frac{8}{63} x^{9} \right) x^{-7} + \left(\frac{4}{7} x^{2} \right) \times 1
}
$$

Simplify each term:

- First term:

$$
- \frac{8}{63} x^{9} \times x^{-7} = - \frac{8}{63} x^{2}
$$

- Second term:

$$
\frac{4}{7} x^{2}
$$

Adding these:

$$
y_p = - \frac{8}{63} x^{2} + \frac{4}{7} x^{2} = \left(- \frac{8}{63} + \frac{36}{63} \right) x^{2} = \frac{28}{63} x^{2} = \frac{4}{9} x^{2}
$$

Final General Solution

Combining the homogeneous and particular solutions:

\[
\boxed{
y(x) = - \frac{C_1}{7} x^{-7} +

Frequently Asked Questions

What is the general approach to solving the differential equation xy'' + 8y' = x8' using variation of parameters?

First, rewrite the differential equation in standard form, find the complementary solution, then apply variation of parameters by assuming particular solutions with variable coefficients, and determine these coefficients to find the particular solution.

How do you identify the homogeneous part of the differential equation xy'' + 8y' = x8'?

The homogeneous equation is obtained by setting the right-hand side to zero: xy'' + 8y' = 0. This helps in finding the complementary solution before applying variation of parameters.

What is the significance of the functions y1 and y2 in the variation of parameters method for this differential equation?

y1 and y2 are the fundamental solutions to the homogeneous equation. They form the basis for constructing the particular solution using variation of parameters.

How do you compute the Wronskian when applying variation of parameters to the equation xy'' + 8y' = x8'?

The Wronskian is computed as W(y1, y2) = y1 y2' - y2 y1'. It is used to find the coefficients in the particular solution formula, ensuring the solutions are linearly independent.

What is the role of the function 8' in the differential equation xy'' + 8y' = x8'?

The term x8' acts as the nonhomogeneous part of the differential equation, representing the forcing function that the particular solution aims to address.

Can you outline the steps to find the particular solution using

variation of parameters for this differential equation?

Yes. First, find the homogeneous solutions y1 and y2. Then, set the particular solution as y_p = u1(x)y1 + u2(x)y2, where u1 and u2 are functions. Use formulas involving the Wronskian to compute u1' and u2', integrate to find u1 and u2, and thus obtain y_p.

Are there any special considerations when applying variation of parameters to equations involving functions like xy'' + 8y' = x8'?

Yes. Ensure that the homogeneous solutions are correctly found, the Wronskian is non-zero, and that the integrals for u1 and u2 are properly evaluated, especially considering the variable coefficients and the presence of x in the nonhomogeneous term.

Additional Resources

Solve The Given Differential Equation By Variation Of Parameters is a classic problem in the study of differential equations, particularly useful for understanding how to handle nonhomogeneous equations that do not lend themselves easily to methods like undetermined coefficients. The process of variation of parameters offers a systematic approach for finding particular solutions to these differential equations, especially when the nonhomogeneous part is complicated or not suitable for simpler methods. This article aims to provide a comprehensive, step-by-step explanation of solving the differential equation $xy'' + 8y' = x^8$ using variation of parameters, along with insights into the underlying concepts, advantages, and limitations of the method.

Understanding the Differential Equation

Before diving into the solution process, it's crucial to understand the nature of the given differential equation:

$$
xy'' + 8y' = x^8
$$

This is a second-order linear differential equation with variable coefficients. Notice the presence of x multiplying the second derivative y'', which indicates it's a form of an Euler-Cauchy type or a variable coefficient differential equation.

Key features:
- It is linear and nonhomogeneous.
- The coefficients are functions of x, complicating the solution.
- The right-hand side x^8 imparts a nonhomogeneous element.

The goal is to find the general solution, which is the sum of the general solution of the associated homogeneous equation and a particular solution that accounts for the nonhomogeneous part.

Step 1: Rewrite the Equation in a Standard Form

To facilitate the solution process, especially for applying variation of parameters, it’s often useful to write the differential equation in a normalized form:

$$
xy'' + 8y' = x^8
$$

Divide through by x (assuming $x \neq 0$):

$$
y'' + \frac{8}{x} y' = x^7
$$

Now, the equation is:

$$
y'' + P(x) y' = Q(x)
$$

where:

$$
P(x) = \frac{8}{x}, \quad Q(x) = x^7
$$

This format clearly indicates a linear second-order differential equation with variable coefficients, suitable for the variation of parameters method.

Step 2: Find the Homogeneous Solution

The associated homogeneous equation is:

$$
y'' + \frac{8}{x} y' = 0
$$

which simplifies to:

$$
y'' + \frac{8}{x} y' = 0
$$

Method: Reduction of order

Let’s set $z = y'$. Then:

\[
z' + \frac{8}{x} z = 0
\]

This is a first-order linear differential equation in \(z\):

\[
z' + \frac{8}{x} z = 0
\]

Solution for \(z\):

This is a separable equation:

\[
\frac{dz}{z} = - \frac{8}{x} dx
\]

Integrate both sides:

\[
\ln |z| = -8 \ln |x| + C_1
\]

\[
z = y' = C x^{-8}
\]

where \(C = e^{C_1}\).

Integrate to find \(y\):

\[
y' = C x^{-8}
\]
\[
y = \int C x^{-8} dx + C_2
\]
\[
y = C \int x^{-8} dx + C_2
\]

Calculate the integral:

\[
\int x^{-8} dx = \frac{x^{-7}}{-7} + K
\]

Therefore, the homogeneous solution:

\[

\boxed{
y_h = A x^{-7} + B
}
\]

where \(A\) and \(B\) are arbitrary constants. The general homogeneous solution is:

\[
\boxed{
y_h = C_1 x^{-7} + C_2
}
\]

Note: The solution shows that the homogeneous equation has solutions involving powers of \(x\), specifically \(x^{-7}\), and a constant. This will be crucial in applying variation of parameters.

Step 3: Set Up for Variation of Parameters

The method of variation of parameters requires two linearly independent solutions of the homogeneous equation. From above, these are:

\[
y_1 = x^{-7}
\]
\[
y_2 = 1
\]

The general homogeneous solution:

\[
y_h = C_1 y_1 + C_2 y_2
\]

The next step is to find particular solutions \(y_p\) of the nonhomogeneous equation by assuming:

\[
y_p = u_1(x) y_1 + u_2(x) y_2
\]

where \(u_1(x)\) and \(u_2(x)\) are functions to be determined.

Key notes:
- The functions \(u_1\) and \(u_2\) are found using the formulas derived from the Wronskian.
- The Wronskian \(W(y_1, y_2)\) is essential in the process.

Step 4: Compute the Wronskian

Calculate the Wronskian W:

$$
W = y_1 y_2' - y_1' y_2
$$

Given:

$$
y_1 = x^{-7}, \quad y_2 = 1
$$

Calculations:

$$
y_1' = -7 x^{-8}
$$
$$
y_2' = 0
$$

Thus:

$$
W = x^{-7} \cdot 0 - (-7 x^{-8}) \cdot 1 = 7 x^{-8}
$$

The Wronskian:

$$
W = 7 x^{-8}
$$

Note: The Wronskian is non-zero for $x \neq 0$, confirming the independence of the solutions.

Step 5: Find u_1' and u_2'

Using the variation of parameters formulas:

$$
u_1' = - \frac{y_2 Q(x)}{W}
$$
$$
u_2' = \frac{y_1 Q(x)}{W}
$$

Plug in known values:

$$
u_1' = - \frac{1 \cdot x^8}{7 x^{-8}} = - \frac{x^8}{7 x^{-8}} = - \frac{x^8}{7} \cdot x^{8} = - \frac{x^{16}}{7}
$$

Similarly,

$$
u_2' = \frac{x^{-7} \cdot x^8}{7 x^{-8}} = \frac{x^{1}}{7 x^{-8}} = \frac{x}{7} \cdot x^{8} = \frac{x^{9}}{7}
$$

Note: Careful algebra confirms the integrals for u_1' and u_2':

$$
u_1' = - \frac{x^{16}}{7}
$$
$$
u_2' = \frac{x^{9}}{7}
$$

Now, integrate to find u_1 and u_2:

$$
u_1 = - \frac{1}{7} \int x^{16} dx = - \frac{1}{7} \cdot \frac{x^{17}}{17} + C_3 = - \frac{x^{17}}{119} + C_3
$$

$$
u_2 = \frac{1}{7} \int x^{9} dx = \frac{1}{7} \cdot \frac{x^{10}}{10} + C_4 = \frac{x^{10}}{70} + C_4
$$

Constants of integration are absorbed into the homogeneous solution, so we focus on the particular part:

$$
y_p = u_1 y_1 + u_2 y_2
$$

which gives:

$$
y_p = - \frac{x^{17}}{119} \cdot x^{-7} + \frac{x^{10}}{70} \cdot 1
$$

Simplify:

$$
y_p = - \frac{x^{17}}{119} \cdot x^{-7} + \frac{x^{10}}{70} = - \frac{x^{10}}{119} + \frac{x^{10}}{70}
$$

Combine the terms:

$$
y_p = x^{10} \left(- \frac{1}{119} + \frac{1}{70} \right)
$$

Find common denominator:

$$
\text{LCD} = 119 \times 70
$$

Calculate:

$$
- \frac{1}{119} + \frac{1}{70} = - \frac{70}{119 \times 70} + \frac{119}{119 \times 70} = \frac{-70 + 119}{119 \times 70} = \frac{49}{119 \times 70}
$$

Simplify numerator and denominator:

\

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