Solve The Initial Value Problem Y" - 6y' + 10y = 0, Y(0) = 1, Y'(0) = 2. =

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Solving initial value problems (IVPs) for differential equations is a fundamental aspect of applied mathematics, engineering, and physics. These problems involve finding a specific solution to a differential equation that satisfies given initial conditions. In this comprehensive guide, we will walk through the process of solving the second-order linear homogeneous differential equation:

$$ Y'' - 6Y' + 10Y = 0 $$

with initial conditions:

$$ Y(0) = 1, \quad Y'(0) = 2 $$

Our goal is to find the explicit function $Y(t)$ that satisfies both the differential equation and the initial conditions.

Understanding the Differential Equation

Before diving into the solution, it's important to interpret the differential equation:

$$ Y'' - 6Y' + 10Y = 0 $$

This is a second-order linear homogeneous differential equation with constant coefficients. Such equations often model physical systems such as mechanical vibrations, electrical circuits, or population dynamics.

Key characteristics:

- Linear: The equation involves derivatives of Y with constant coefficients.
- Homogeneous: The right-hand side is zero, simplifying the solution process.
- Second-order: Involving Y'', requiring two initial conditions for a unique solution.

Step 1: Formulate the Characteristic Equation

To solve the differential equation, the standard approach involves assuming a solution of the form:

$$ Y(t) = e^{rt} $$

where r is a constant to be determined. Substituting this into the differential equation yields the characteristic (or auxiliary) equation.

Derivation:

1. Compute derivatives:

$$ Y' = r e^{rt} $$
$$ Y'' = r^2 e^{rt} $$

2. Substitute into the differential equation:

$$ r^2 e^{rt} - 6 r e^{rt} + 10 e^{rt} = 0 $$

3. Factor out e^{rt}, which is never zero:

$$ e^{rt} (r^2 - 6 r + 10) = 0 $$

4. The characteristic equation:

$$ r^2 - 6 r + 10 = 0 $$

Step 2: Solve the Characteristic Equation

The quadratic characteristic equation:

$$ r^2 - 6 r + 10 = 0 $$

can be solved using the quadratic formula:

$$ r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} $$

where $a=1$, $b=-6$, and $c=10$.

Calculations:

1. Compute discriminant:

$$ \Delta = b^2 - 4ac = (-6)^2 - 4 \times 1 \times 10 = 36 - 40 = -4 $$

2. Since the discriminant is negative, roots are complex conjugates:

$$ r = \frac{6 \pm \sqrt{-4}}{2} $$

3. Simplify:

$$ r = \frac{6 \pm 2i}{2} = 3 \pm i $$

Result:

$$ r_1 = 3 + i $$
$$ r_2 = 3 - i $$

These complex roots indicate the solution involves exponential functions combined with sinusoidal functions.

Step 3: Write the General Solution

For roots of the form $r = \alpha \pm \beta i$, the general solution to the differential equation is:

$$ Y(t) = e^{\alpha t} (C_1 \cos \beta t + C_2 \sin \beta t) $$

In our case:

$$ \alpha = 3 $$
$$ \beta = 1 $$

Thus, the general solution:

$$ Y(t) = e^{3 t} (C_1 \cos t + C_2 \sin t) $$

where C_1 and C_2 are constants determined by initial conditions.

Step 4: Apply Initial Conditions to Determine Constants

Given initial conditions:

$$ Y(0) = 1 $$
$$ Y'(0) = 2 $$

we substitute $t=0$ into the general solution and its derivative to find C_1 and C_2.

Compute $Y(0)$:

$$ Y(0) = e^{0} (C_1 \cos 0 + C_2 \sin 0) = 1 \times (C_1 \times 1 + C_2 \times 0) = C_1 $$

So,

$$ C_1 = 1 $$

Compute $Y'(t)$:

Differentiate $Y(t)$:

$$
Y(t) = e^{3 t} (C_1 \cos t + C_2 \sin t)
$$

Using the product rule:

\[
Y'(t) = e^{3 t} \times 3 (C_1 \cos t + C_2 \sin t) + e^{3 t} \times (- C_1

$$
\sin t + C_2 \cos t)
$$

Simplify:

$$
Y'(t) = e^{3 t} [3 C_1 \cos t + 3 C_2 \sin t - C_1 \sin t + C_2 \cos t]
$$

At $t=0$:

$$
Y'(0) = e^{0} [3 C_1 \times 1 + 3 C_2 \times 0 - C_1 \times 0 + C_2 \times 1] = (3 C_1 + C_2)
$$

Given $Y'(0) = 2$, and $C_1 = 1$:

$$
2 = 3 \times 1 + C_2 \Rightarrow C_2 = 2 - 3 = -1
$$

Step 5: Write the Final Solution

Plugging the constants back into the general solution:

$$ Y(t) = e^{3 t} (1 \times \cos t - 1 \times \sin t) $$

or more neatly:

$$ \boxed{
Y(t) = e^{3 t} (\cos t - \sin t)
} $$

This is the unique solution to the initial value problem.

Summary and Key Takeaways

- The differential equation was identified as a second-order linear homogeneous with constant coefficients.
- The characteristic equation yielded complex roots $3 \pm i$, leading to a solution involving exponential and sinusoidal functions.
- The general solution is:

$$ Y(t) = e^{3 t} (C_1 \cos t + C_2 \sin t) $$

- Using initial conditions, we determined:

$$ C_1 = 1, \quad C_2 = -1 $$

- The specific solution satisfying the initial conditions:

$$ Y(t) = e^{3 t} (\cos t - \sin t) $$

Additional Insights and Applications

Understanding how to solve such differential equations is crucial in various fields:

- Mechanical Engineering: Modeling damped oscillations and vibrations.
- Electrical Engineering: Analyzing RLC circuits.
- Physics: Describing harmonic oscillators and wave phenomena.
- Mathematics: Developing intuition for complex roots and their implications on solution behavior.

Behavior of the solution:

- The exponential term e^{3t} indicates growth over time.
- The sine and cosine terms introduce oscillatory behavior.
- The combination suggests a system with exponential growth modulated by oscillations.

Conclusion

By systematically approaching the initial value problem, we used characteristic equations to find the general solution, then applied initial conditions to find the particular solution. Mastering these steps allows for solving a wide array of second-order differential equations encountered in scientific and engineering contexts. This process underscores the importance of understanding roots of characteristic equations, the structure of solutions, and the application of initial conditions to tailor solutions to specific problems.

If you want to deepen your understanding, consider exploring:

- Non-homogeneous differential equations.
- Differential equations with variable coefficients.
- Numerical methods for solving differential equations when closed-form solutions are infeasible.

By mastering these techniques, you'll be well-equipped to tackle complex initial value problems across disciplines.

Frequently Asked Questions

What is the general solution to the differential equation Y'' - 6Y' + 10Y = 0?

The characteristic equation is $r^2 - 6r + 10 = 0$, which has roots $r = 3 \pm i$. Therefore, the general solution is $Y(t) = e^{3t} (C_1 \cos t + C_2 \sin t)$.

How do we determine the particular solution given the initial conditions Y(0) = 1 and Y'(0) = 2?

We substitute $t = 0$ into the general solution and its derivative to solve for constants C_1 and C_2: $Y(0) = C_1 = 1$, and $Y'(0) = 3C_1 + C_2 = 2$. Using these, we find $C_1 = 1$ and $C_2 = -1$.

What is the explicit solution to the initial value problem?

Using the calculated constants, the solution is $Y(t) = e^{3t} (\cos t - \sin t)$.

What is the significance of the roots being complex in solving the differential equation?

Complex roots indicate oscillatory behavior in the solution, with exponential growth or decay depending on the real part. In this case, the solution exhibits exponential growth combined with sinusoidal oscillations.

How does the initial condition Y'(0) = 2 influence the determination of the constants in the solution?

The initial derivative condition provides an equation involving C_1 and C_2, allowing us to solve for both constants in conjunction with $Y(0) = 1$, ensuring a unique solution to the IVP.

Additional Resources

Solve The Initial Value Problem $Y'' - 6Y' + 10Y = 0$, $Y(0) = 1$, $Y'(0) = 2$ is a classic second-order linear differential equation with constant coefficients. Such problems are fundamental in the study of differential equations because they model a wide range of physical phenomena, from mechanical vibrations to electrical circuits. Solving these initial value problems (IVPs) involves finding a specific solution that satisfies both the differential equation and the initial conditions, providing a complete description of the system's behavior over time. This article aims to explore the methods, solutions, and implications of this particular IVP, emphasizing the mathematical techniques involved and their applications.

Understanding the Differential Equation

Formulation and Significance

The differential equation in question,

$$Y'' - 6Y' + 10Y = 0,$$

is a homogeneous linear second-order differential equation with constant coefficients. The general form for such an equation is:

$$aY'' + bY' + cY = 0,$$

where a, b, c are constants. Here, $a = 1$, $b = -6$, and $c = 10$.

This type of differential equation appears in many physical systems, such as damped harmonic oscillators (without damping if the damping coefficient is zero), electrical circuits involving inductors and capacitors, and mechanical systems with mass and stiffness. Its solutions reveal the nature of the system's response—whether it oscillates, decays, or grows over time.

Initial Conditions

The initial conditions are:

$$Y(0) = 1, \quad Y'(0) = 2.$$

These specify the initial position and initial velocity of the system, respectively. Applying these conditions ensures that the particular solution models the actual physical scenario under consideration.

Methodology for Solving the Differential Equation

Characteristic Equation Approach

The most straightforward method to solve homogeneous linear differential equations with constant coefficients is to assume solutions of the form:

$$Y(t) = e^{rt},$$

where r is a constant to be determined. Substituting this into the differential equation yields the characteristic equation:

$$r^2 - 6r + 10 = 0.$$

Solving this quadratic will give us the roots that determine the form of the general solution.

Solving the Characteristic Equation

The quadratic equation:

$$
r^2 - 6r + 10 = 0,
$$

can be solved using the quadratic formula:

$$
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
$$

Plugging in the values:

$$
r = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 10}}{2} = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm \sqrt{-4}}{2}.
$$

Since the discriminant is negative (-4), the roots are complex conjugates:

$$
r = \frac{6 \pm 2i}{2} = 3 \pm i.
$$

The roots are $r_1 = 3 + i$, and $r_2 = 3 - i$.

General Solution of the Differential Equation

Form of the Solution

When roots are complex conjugates $r = \alpha \pm \beta i$, the general solution is:

$$
Y(t) = e^{\alpha t} \left(C_1 \cos \beta t + C_2 \sin \beta t \right),
$$

where C_1 and C_2 are arbitrary constants determined by initial conditions.

For our roots:

$$
\alpha = 3, \quad \beta = 1,

$$

thus,

$$
Y(t) = e^{3t} \left(C_1 \cos t + C_2 \sin t \right).
$$

Explicit Form of the General Solution

The general solution becomes:

$$
Y(t) = e^{3t} \left(C_1 \cos t + C_2 \sin t \right).
$$

This solution describes oscillatory behavior modulated by exponential growth, characteristic of systems with complex conjugate roots with positive real parts.

Applying Initial Conditions to Find Particular Solution

Calculating $Y(0)$

Substitute $t=0$:

$$
Y(0) = e^{0} \left(C_1 \cos 0 + C_2 \sin 0 \right) = 1 \times (C_1 \times 1 + C_2 \times 0) = C_1.
$$

Given $Y(0) = 1$, we have:

$$
C_1 = 1.
$$

Calculating $Y'(t)$

Derivative of $Y(t)$:

$$
Y'(t) = \frac{d}{dt} \left[e^{3t} (C_1 \cos t + C_2 \sin t) \right].
$$

Applying product rule:

$$
Y'(t) = e^{3t} \times 3 (C_1 \cos t + C_2 \sin t) + e^{3t} \times (- C_1 \sin t + C_2 \cos t).
$$

Simplify:

$$
Y'(t) = e^{3t} \left[3 C_1 \cos t + 3 C_2 \sin t - C_1 \sin t + C_2 \cos t \right].
$$

At $t=0$:

$$
Y'(0) = e^{0} \left[3 C_1 \times 1 + 3 C_2 \times 0 - C_1 \times 0 + C_2 \times 1 \right] = (3 C_1 + C_2).
$$

Given $Y'(0) = 2$, and $C_1 = 1$:

$$
2 = 3 \times 1 + C_2 \Rightarrow C_2 = 2 - 3 = -1.
$$

Final Solution

Putting it all together, the particular solution satisfying the initial conditions is:

$$
Y(t) = e^{3t} \left(\cos t - \sin t \right).
$$

This solution provides a complete description of the system's behavior over time, influenced by exponential growth and oscillations.

Features and Analysis of the Solution

Behavior Over Time

- Exponential Growth: Since the exponential term e^{3t} has a positive exponent, the solution demonstrates growth as $t \to \infty$. This indicates an unstable system if interpreted physically, such as an amplifier with positive feedback.

- Oscillations: The sinusoidal components $\cos t$ and $\sin t$ introduce oscillatory behavior, characteristic of systems with complex roots.

- Amplitude Modulation: The combination results in oscillations whose amplitude increases exponentially, a typical feature of systems with roots having positive real parts.

Graphical Interpretation

Plotting $Y(t)$ would reveal an oscillatory pattern with amplitudes increasing exponentially. The oscillations cross zero periodically, with peaks and troughs growing larger over time.

Physical Implications

- Unstable System: The exponential growth suggests the system is not stable—small disturbances grow over time.

- Design Considerations: In engineering, such solutions highlight the need for damping or stabilizing mechanisms to prevent runaway behavior.

Pros and Cons of the Solution Approach

Pros:

- Analytical Clarity: The characteristic equation method provides explicit formulas for solutions, aiding understanding.

- Initial Condition Integration: Applying initial conditions straightforwardly yields specific solutions.

- Versatility: The approach applies to a broad class of linear constant-coefficient differential equations.

Cons:

- Limited to Linear Homogeneous Equations: Nonlinear or nonhomogeneous equations require different methods.

- Complex Roots Interpretation: When roots are complex, understanding the physical meaning of oscillations and growth can be challenging.

- Behavioral Limitations: The method does not directly address stability or control issues—additional analysis is often necessary.

Conclusion

The initial value problem $Y'' - 6Y' + 10Y = 0$, with initial conditions $Y(0) = 1$ and $Y'(0) = 2$, exemplifies the power and elegance of solving second-order linear differential equations with constant coefficients. Through characteristic equation methods, we derived the general solution involving exponential and sinusoidal functions, then applied initial conditions to find the particular solution:

$$
\boxed{
Y(t) = e^{3t} (\cos t - \sin t).
}
$$

This solution not only satisfies the mathematical requirements but also provides insights into the dynamic behavior of the modeled

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