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Understanding the relationship between the probability of combined events and their individual probabilities is fundamental in probability theory. A key formula in this context is the probability of the union of two events, denoted as $\Pr(A \cup B)$. This formula expresses how the probabilities of individual events A and B, along with their intersection, combine to give the probability that either event A or event B occurs. In this article, we will explore how to derive the formula:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

using the axioms of probability. This derivation provides a rigorous foundation rooted in the fundamental principles of probability theory, offering both theoretical insight and practical applications.

Understanding the Axioms of Probability

Before diving into the derivation, it’s crucial to understand the three core axioms of probability established by Andrey Kolmogorov, which form the foundation of modern probability theory. These axioms are:

The Axiom of Non-Negativity

- For any event A, $\Pr(A) \geq 0$.

The Axiom of Normalization

- The probability of the certain event (the sample space S) is 1: Pr(S) = 1.

The Axiom of Countable Additivity

- For any countable sequence of mutually exclusive events A□, A□, A□, …, we have:

Pr(□□ A□) = □□ Pr(A□).

These axioms provide the rules for manipulating probabilities and serve as the basis for deriving more complex formulas, such as the probability of unions and intersections of events.

Foundation: Events and Their Probabilities

In probability theory, events are subsets of a sample space S, which contains all possible outcomes of a random experiment. The probability measure Pr assigns a real number between 0 and 1 to each event, satisfying the axioms above.

When considering two events A and B within the sample space, their probabilities and relationships — such as intersection and union — are central to understanding combined events:

- A □ B: both A and B occur.
- A □ B: either A or B (or both) occur.

The goal is to express Pr(A □ B) in terms of Pr(A), Pr(B), and Pr(A □ B).

Deriving the Formula Using Axioms of Probability

The derivation hinges on the principle of additivity for mutually exclusive events and the properties of intersections and unions.

Step 1: Recognize the Relationship of Events

Note that the union of A and B can be partitioned into two mutually exclusive parts:

- The part where A occurs but B does not: A \ (A □ B).
- The part where B occurs but A does not: B \ (A □ B).
- The part where both A and B occur: A □ B.

Mathematically:

A □ B = [A \ (A □ B)] □ [B \ (A □ B)] □ (A □ B)

This partition allows us to express the union as a disjoint union of events.

Step 2: Express Pr(A □ B) as Sum of Probabilities of Disjoint Events

Since the three events [A \ (A □ B)], [B \ (A □ B)], and (A □ B) are mutually exclusive, by the axiom of countable additivity, we have:

Pr(A □ B) = Pr([A \ (A □ B)]) + Pr([B \ (A □ B)]) + Pr(A □ B)

Step 3: Express Probabilities of the Set Differences

Using set algebra and the properties of probability, we can write:

- Pr([A \ (A □ B)]) = Pr(A) − Pr(A □ B)

- Pr([B \ (A □ B)]) = Pr(B) − Pr(A □ B)

This follows because:

Pr(A) = Pr([A \ (A □ B)]) + Pr(A □ B)

Similarly for B.

Step 4: Combine the Expressions

Substituting into the expression for Pr(A □ B):

Pr(A □ B) = [Pr(A) − Pr(A □ B)] + [Pr(B) − Pr(A □ B)] + Pr(A □ B)

Simplify the sum:

Pr(A □ B) = Pr(A) + Pr(B) − Pr(A □ B)

This completes the derivation, showing that:

Pr(A □ B) = Pr(A) + Pr(B) − Pr(A □ B)

Significance of the Formula in Probability Theory

The formula $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$ is fundamental in probability theory for several reasons:

- **Prevents Double Counting:** When summing probabilities of overlapping events, subtracting the intersection avoids counting the shared outcomes twice.
- **Foundation for Inclusion-Exclusion Principle:** This formula is the basis for the inclusion-exclusion principle, which generalizes to more than two events.
- **Supports Conditional Probability:** The understanding of intersections is crucial in calculating conditional probabilities, which depend on the likelihood of events given others.
- **Application in Real-World Problems:** Whether in statistics, finance, or engineering, this formula helps in calculating probabilities of combined events where dependencies exist.

Applications and Examples

To solidify understanding, here are practical examples illustrating how to apply the formula:

Example 1: Drawing Cards

Suppose you draw a card from a standard deck of 52 cards:

- Event A: Drawing a King.
- Event B: Drawing a Heart.

Calculate $\Pr(A \cup B)$.

- Pr(A): Probability of drawing a King = 4/52 = 1/13.
- Pr(B): Probability of drawing a Heart = 13/52 = 1/4.
- $\Pr(A \cap B)$: Probability of drawing the King of Hearts = 1/52.

Applying the formula:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = (1/13) + (1/4) - (1/52)$$

Find common denominator (52):

$$(4/52) + (13/52) - (1/52) = (4 + 13 - 1)/52 = 16/52 = 4/13$$

Thus, the probability of drawing a King or a Heart is 4/13.

Example 2: Medical Testing

In a medical screening test:

- Event A: Test indicates disease presence.
- Event B: Patient actually has the disease.

Given:

- $Pr(A)$: Probability test indicates disease = 0.05.
- $Pr(B)$: Probability of actual disease in the population = 0.01.
- $Pr(A \cap B)$: Probability test indicates disease when disease is present = 0.009.

Calculate the probability that the test indicates disease or the patient actually has the disease:

$$Pr(A \cup B) = 0.05 + 0.01 - 0.009 = 0.051$$

This helps in understanding the combined likelihood, accounting for overlaps.

Conclusion

Using the axioms of probability, particularly the axioms of non-negativity, normalization, and countable additivity, we have rigorously derived the fundamental formula for the probability of the union of two events:

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

This formula is central to probability theory, ensuring accurate computation of combined event probabilities without double counting overlaps. Whether in theoretical studies or practical applications, understanding this derivation enhances your grasp of probability concepts, enabling more precise analysis of complex probabilistic scenarios.

By mastering this derivation, you lay the groundwork for exploring more advanced topics such as the inclusion-exclusion principle, conditional probability, and Bayesian inference, all of which are essential tools in statistics, data science, and decision-making processes.

Keywords: probability axioms, union of events, intersection, probability derivation, probability theory, inclusion-exclusion principle, conditional probability, probability formula, set theory in probability

Frequently Asked Questions

What is the axiom of probability that relates to the union of two events?

The axiom states that for any two events A and B, the probability of their union is given by $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

How can the axioms of probability be used to derive the formula for $Pr(A \cup B)$?

By applying the additivity axiom and recognizing that the intersection is counted twice when summing Pr(A) and Pr(B), we subtract $Pr(A \cap B)$ to correct the count, leading to $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

Why do we subtract $Pr(A \cap B)$ when calculating $Pr(A \cup B)$?

Because the intersection $Pr(A \cap B)$ is included in both Pr(A) and Pr(B), subtracting it prevents double-counting in the union probability.

Can the formula $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$ be applied to any events?

Yes, this formula holds for any events A and B within a probability space, regardless of whether they are independent or dependent.

How does the inclusion-exclusion principle relate to the axiom-based proof of Pr(A B)?

The inclusion-exclusion principle formalizes the idea that the probability of the union is the sum of individual probabilities minus the probability of their intersection, which directly follows from probability axioms.

What assumptions are necessary to use the formula Pr(A B) = Pr(A) + Pr(B) - Pr(A B)?

The primary assumption is that probabilities are well-defined within a probability space and that the events are measurable; no independence assumption is required.

How do the probability axioms ensure the validity of the union formula?

The axioms, especially additivity and normalization, guarantee that the probabilities behave consistently, allowing the derivation of the union formula through logical combination of event probabilities.

Additional Resources

Use The Axioms Of Probability To Show That Pr(A B) = Pr(A) + Pr(B) - Pr(A B)

Introduction

Probability theory is a foundational pillar of modern mathematics, underpinning fields as diverse as statistics, physics, computer science, economics, and engineering. Its robust mathematical framework is built upon a set of axioms that formalize the intuitive notions of chance and likelihood. Among these axioms, the principles governing events' probabilities and their combined occurrences are particularly crucial, especially when analyzing the union and intersection of events.

A central result in probability theory is the Addition Rule, which expresses the probability of the union of two events in terms of their individual probabilities and the probability of their intersection. The formula:

$$
\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)
$$

is fundamental in calculating combined probabilities and understanding how events overlap.

This article provides an in-depth exploration of how the axioms of probability lead to this important identity. We will begin by establishing the axioms, then proceed to develop the proof step-by-step, illustrating key concepts along the way.

The Axioms of Probability

To understand the derivation, it is essential first to recall the three axioms proposed by Andrey Kolmogorov, which form the bedrock of modern probability theory:

1. Non-negativity Axiom:
For any event A,
$$
\Pr(A) \geq 0
$$
2. Normalization Axiom:

The probability of the entire sample space S is 1:

$$
\Pr(S) = 1
$$

3. Countable Additivity (□-additivity):

For any countable sequence of mutually exclusive events $A_1, A_2, \ldots,$

$$
\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)
$$

The second axiom establishes a scale for probability, while the third axiom ensures that probabilities are additive over disjoint events.

Fundamental Concepts: Events, Union, and Intersection

Before delving into the proof, it is helpful to clarify the notions of union and intersection of events:

- Union ($A \cup B$): the event that either A occurs, or B occurs, or both.
- Intersection ($A \cap B$): the event that both A and B occur simultaneously.

Visualizing events with Venn diagrams often aids in comprehending their relationships, particularly overlaps.

Deriving the Addition Rule Using Probability Axioms

Step 1: Recognize the Relationship Between $A \cup B$, A, B, and $A \cap B$

The union $A \cup B$ can be partitioned into two mutually exclusive events:

- The event $A \setminus B$: occurs if A occurs and B does not.
- The event $B \setminus A$: occurs if B occurs and A does not.
- The event $A \cap B$: occurs if both A and B occur simultaneously.

Formally:

$$
A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)
$$

with all three events mutually exclusive.

Step 2: Express $\Pr(A \cup B)$ in Terms of These Disjoint Events

By the axioms, the probability of a union of mutually exclusive events is the sum of their probabilities:

$$
\Pr(A \cup B) = \Pr(A \setminus B) + \Pr(B \setminus A) + \Pr(A \cap B)
$$

Similarly, the probabilities of the disjoint events relate to the probabilities of A and B:

- $\Pr(A) = \Pr(A \setminus B) + \Pr(A \cap B)$
- $\Pr(B) = \Pr(B \setminus A) + \Pr(A \cap B)$

Step 3: Express $\Pr(A \setminus B)$ and $\Pr(B \setminus A)$

From the above:

$$
\Pr(A \setminus B) = \Pr(A) - \Pr(A \cap B)
$$

$$
\Pr(B \setminus A) = \Pr(B) - \Pr(A \cap B)
$$

Since these two events are mutually exclusive and disjoint with $A \cap B$, their sum is:

$$
\Pr(A \setminus B) + \Pr(B \setminus A) = \Pr(A) + \Pr(B) - 2\Pr(A \cap B)
$$

Step 4: Assemble the Final Expression

Now, substitute back into the expression for $\Pr(A \cup B)$:

$$
\Pr(A \cup B) = [\Pr(A) - \Pr(A \cap B)] + [\Pr(B) - \Pr(A \cap B)] + \Pr(A \cap B)
$$

Simplify:

$$
\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)
$$

This completes the derivation based solely on the axioms and the properties of mutually exclusive events.

Intuitive Explanation

The formula essentially corrects for double-counting the intersection. When adding $\Pr(A)$ and $\Pr(B)$, the overlapping part $\Pr(A \cap B)$ gets counted twice. Subtracting $\Pr(A \cap B)$ ensures that the overlapping probability is only counted once, providing an accurate measure of the probability that either A or B occurs.

Formal Proof Summary

Step	Description	Mathematical Expression
1	Partition $A \cup B$ into disjoint events	$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$
2	Write $\Pr(A \cup B)$ as sum over disjoint events	$\Pr(A \cup B) = \Pr(A \setminus B) + \Pr(B \setminus A) + \Pr(A \cap B)$
3	Express $\Pr(A)$ and $\Pr(B)$ in terms of parts	$\Pr(A) = \Pr(A \setminus B) + \Pr(A \cap B)$, and similarly for B
4	Find $\Pr(A \setminus B)$ and $\Pr(B \setminus A)$	$\Pr(A \setminus B) = \Pr(A) - \Pr(A \cap B)$, etc.
5	Substitute into the union probability	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Broader Implications and Applications

Understanding this derivation is not merely an academic exercise; it underpins many practical applications:

- Risk assessment: Calculating the probability of at least one of multiple risk factors occurring.

- Statistics: Computing combined probabilities in contingency tables.
- Machine Learning: Estimating event overlaps in classification problems.
- Engineering: Reliability analysis involving overlapping component failures.

Moreover, the addition rule forms a stepping stone toward more complex probability identities, including the inclusion-exclusion principle for multiple events.

Extending to Multiple Events

While this discussion focuses on two events, the principle generalizes to multiple events via the inclusion-exclusion principle:

\[
\Pr\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n \Pr(A_i) - \sum_{i