Y''-y'-6y=0 Y(0)=1, Y'(0)=-1Using Laplace Transform To Soilve Initial Value

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Solving differential equations is a fundamental aspect of mathematical analysis, especially in engineering, physics, and applied sciences. One powerful method for solving linear ordinary differential equations (ODEs) with initial conditions is the Laplace Transform. In this article, we explore how to use the Laplace Transform to solve the second-order differential equation:

Understanding the Differential Equation and Initial Conditions

The Differential Equation

The differential equation under consideration is:

$$y'' - y' - 6y = 0$$

This is a linear second-order homogeneous differential equation with constant coefficients.

Initial Conditions

The initial conditions provided are:

$$y(0) = 1$$

$$y'(0) = -1$$

These initial conditions are crucial for finding a unique solution to the differential equation using the Laplace Transform method.

Applying the Laplace Transform

Step 1: Take the Laplace Transform of Both Sides

Recall that the Laplace Transform of a function $y(t)$ is defined as:

$$L\{y(t)\} = Y(s) = \int_0^\infty e^{-st} y(t)\, dt$$

Applying the Laplace Transform to the differential equation:

$$L\{y'' - y' - 6y\} = 0$$

Using linearity:

$$L\{y''\} - L\{y'\} - 6L\{y\} = 0$$

Step 2: Use the Properties of Laplace Transform

The Laplace Transform of derivatives are:

$L\{y'\} = sY(s) - y(0)$

$L\{Y''\} = s^2Y(s) - s\,y(0) - y'(0)$

Substituting the initial conditions:

$Y(0) = 1$

$Y'(0) = -1$

We get:

$L\{Y''\} = s^2Y(s) - s1 - (-1) = s^2Y(s) - s + 1$

Similarly,

$L\{y'\} = sY(s) - y(0) = sY(s) - 1$

And,

$L\{y\} = Y(s)$

Step 3: Write the Transformed Equation

Putting all into the transformed differential equation:

$[s^2Y(s) - s + 1] - [sY(s) - 1] - 6Y(s) = 0$

Simplify:

$s^2Y(s) - s + 1 - sY(s) + 1 - 6Y(s) = 0$

Combine like terms:

$(s^2Y(s) - sY(s) - 6Y(s)) + (-s + 1 + 1) = 0$

Factor Y(s):

$Y(s)(s^2 - s - 6) + (-s + 2) = 0$

Solving for Y(s)

Step 4: Isolate Y(s)

Rearranged:

$Y(s)(s^2 - s - 6) = s - 2$

Therefore,

$Y(s) = (s - 2) / (s^2 - s - 6)$

Step 5: Factor the Denominator

Factor the quadratic:

$s^2 - s - 6 = (s - 3)(s + 2)$

Thus,

$Y(s) = (s - 2) / [(s - 3)(s + 2)]$

Inverse Laplace Transform to Find y(t)

Step 6: Partial Fraction Decomposition

Express Y(s) as partial fractions:

$(s - 2) / [(s - 3)(s + 2)] = A / (s - 3) + B / (s + 2)$

Multiply both sides by (s - 3)(s + 2):

$s - 2 = A(s + 2) + B(s - 3)$

Set up equations to solve for A and B:

For s = 3:

$3 - 2 = A(3 + 2) + B(3 - 3)$

$1 = A(5) + B(0)$

$A = 1/5$

For s = -2:

$-2 - 2 = A(-2 + 2) + B(-2 - 3)$

$-4 = A(0) + B(-5)$

$B = 4/5$

Step 7: Write the Partial Fractions

$Y(s) = (1/5) / (s - 3) + (4/5) / (s + 2)$

Step 8: Find the Inverse Laplace Transform

Recall:

$L^{-1} \{1 / (s - a)\} = e^{a t}$

So,

$y(t) = (1/5) e^{3t} + (4/5) e^{-2t}$

This is the particular solution satisfying the initial conditions.

Final Solution and Interpretation

The solution to the initial value problem:

$y'' - y' - 6y = 0$

with initial conditions $y(0) = 1$ and $y'(0) = -1$, is:

```
y(t) = (1/5) e^{3t} + (4/5) e^{-2t}
```

This solution describes the behavior of the system modeled by the differential equation, often representing physical phenomena such as oscillations, vibrations, or electrical circuits.

Benefits of Using Laplace Transform in Differential Equations

Ease of Handling Initial Conditions

The Laplace Transform naturally incorporates initial conditions, simplifying the process of solving initial value problems.

Reduction of Differential Equations to Algebraic Equations

Transforming derivatives into algebraic expressions makes solving complex differential equations more straightforward.

Applicability to Non-Homogeneous Equations

The Laplace Transform is equally effective for non-homogeneous equations with forcing functions.

Summary

Using the Laplace Transform to solve differential equations involves transforming the original problem into an algebraic equation in the s-domain, solving for the transformed function, and then applying the inverse Laplace Transform to find the solution in the time domain. For the differential equation $y'' - y' - 6y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = -1$, this process yields a clear analytical solution:

```
y(t) = (1/5) e^{3t} + (4/5) e^{-2t}
```

This method not only simplifies the process but also provides a systematic approach to solving a wide class of initial value problems in differential equations.

Additional Resources

- Understanding Laplace Transform properties and tables
- Solving differential equations with non-zero initial conditions
- Applications of Laplace Transform in engineering and physics

By mastering the use of the Laplace Transform, students and professionals can efficiently tackle complex differential equations, ensuring accurate and insightful solutions to real-world problems.

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $y'' - y'' - 6y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = -1$.

How do you apply Laplace Transform to solve the differential equation?

Apply Laplace Transform to both sides of the differential equation, convert derivatives into algebraic expressions involving Laplace variable s, and then solve for Y(s).

What are the steps to find the solution y(t) using Laplace Transform?

First, take the Laplace Transform of the differential equation, substitute initial conditions, solve for Y(s), and then find the inverse Laplace Transform to get y(t).

How are initial conditions used in the Laplace Transform method?

Initial conditions $y(0) = 1$ and $y'(0) = -1$ are used to evaluate the Laplace Transforms of derivatives, replacing derivative terms with algebraic expressions involving s and initial values.

What is the Laplace Transform of the derivatives y'' and y'?

The Laplace Transform of y'' is $s^2 Y(s) - s\,y(0) - y'(0)$, and for y' it is $s\,Y(s) - y(0)$.

What is the final solution for y(t) after applying Laplace Transform and inverse transform?

After solving for Y(s) and applying the inverse Laplace Transform, the solution is $y(t) = \cos(\sqrt{6}\, t) + (1/\sqrt{6}) \sin(\sqrt{6}\, t)$.

Why is the Laplace Transform method advantageous for solving initial value problems like this?

It simplifies the process by converting differential equations into algebraic equations, making it easier to incorporate initial conditions and find solutions systematically.

Additional Resources

Using Laplace Transform To Solve Initial Value Problems: A Step-by-Step Guide

When faced with differential equations, especially those with initial conditions, the Laplace transform to solve initial value problems offers a powerful and systematic approach. This technique simplifies differential equations into algebraic equations, making it easier to find solutions, particularly for linear ordinary differential equations with constant coefficients. In this guide, we will explore how to apply the Laplace transform to solve the initial value problem:

$$ y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1 $$

demonstrating each step in detail, with explanations designed for students and professionals alike.

Understanding the Differential Equation and Initial Conditions

Before applying the Laplace transform, let's clarify the problem:

- Differential Equation: $y'' - y' - 6y = 0$
- Initial Conditions: $y(0) = 1$, $y'(0) = -1$

This is a second-order linear homogeneous differential equation with constant coefficients. The initial conditions specify the value of the solution and its first derivative at $t=0$.

Why Use the Laplace Transform?

The Laplace transform converts a differential equation in the time domain into an algebraic equation in the complex frequency domain (the s-domain). This process simplifies the problem because:

- Derivatives become algebraic multiplications by s.
- Initial conditions are incorporated naturally.
- Once algebraic, the equation can be solved for the Laplace transform of the unknown function, $Y(s)$.
- The inverse Laplace transform then yields the solution in the original domain.

Step 1: Recall the Laplace Transform Definitions

The key Laplace transform pairs and properties we'll use:

- $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\, dt$
- $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$
- $\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$

Step 2: Apply the Laplace Transform to the Differential Equation

Transform each term:

1. $y''(t) \rightarrow s^2Y(s) - s\, y(0) - y'(0)$
2. $y'(t) \rightarrow sY(s) - y(0)$
3. $y(t) \rightarrow Y(s)$

Plugging into the differential equation:

$$
s^2Y(s) - s\, y(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) = 0
$$

Now, substitute the known initial conditions $y(0) = 1$, $y'(0) = -1$:

$$
s^2Y(s) - s \cdot 1 - (-1) - sY(s) + 1 - 6Y(s) = 0
$$

Simplify:

$$
s^2Y(s) - s + 1 - sY(s) + 1 - 6Y(s) = 0
$$

Combine like terms:

$$
(s^2Y(s) - sY(s) - 6Y(s)) + (-s + 1 + 1) = 0
$$

Factor $Y(s)$:

$$
Y(s)(s^2 - s - 6) + (-s + 2) = 0
$$

Rearranged:

$$
Y(s)(s^2 - s - 6) = s - 2
$$

Step 3: Solve for $Y(s)$

Divide both sides by $s^2 - s - 6$:

$$
Y(s) = \frac{s - 2}{s^2 - s - 6}
$$

Factor the denominator:

$$
s^2 - s - 6 = (s - 3)(s + 2)
$$

Thus:

$$
Y(s) = \frac{s - 2}{(s - 3)(s + 2)}
$$

Step 4: Partial Fraction Decomposition

To find the inverse Laplace transform, decompose $Y(s)$:

$$

$$
\frac{s - 2}{(s - 3)(s + 2)} = \frac{A}{s - 3} + \frac{B}{s + 2}
$$

Multiply both sides by $(s - 3)(s + 2)$:

$$
s - 2 = A(s + 2) + B(s - 3)
$$

Set up the equations by equating coefficients:

- For s: $1 = A + B$
- For constant term: $-2 = 2A - 3B$

Solve for A and B:

From the first:

$$
A + B = 1 \Rightarrow B = 1 - A
$$

Substitute into second:

$$
-2 = 2A - 3(1 - A) = 2A - 3 + 3A = 5A - 3
$$

Solve for A:

$$
5A = 1 \Rightarrow A = \frac{1}{5}
$$

Then:

$$
B = 1 - \frac{1}{5} = \frac{4}{5}
$$

Hence:

$$
Y(s) = \frac{1/5}{s - 3} + \frac{4/5}{s + 2}
$$

Step 5: Find the Inverse Laplace Transform

Recall:

- $\mathcal{L}^{-1}\left\{\frac{1}{s - a}\right\} = e^{at}$

Apply this to each term:

$$
y(t) = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t}
$$

Final Solution:

$$
\boxed{
\textbf{Solution:} \quad y(t) = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t}
}
$$

This function satisfies the initial value problem, as verified by substituting back into the original differential equation and initial conditions.

Summary and Key Takeaways

- The Laplace transform simplifies solving linear differential equations with initial conditions by converting derivatives into algebraic expressions.
- Correctly applying the transform requires familiarity with the properties and initial condition incorporation.
- Partial fraction decomposition is often necessary to invert Laplace transforms into elementary functions.
- The method is especially advantageous for nonhomogeneous equations or those with forcing functions, but it is equally effective for homogeneous equations as demonstrated.

Additional Tips for Using Laplace Transforms

- Always verify your initial conditions are correctly incorporated.
- Factor denominators to facilitate partial fractions.
- Use tables of Laplace transforms for quick inversion.
- Remember to check the domain of convergence for your transforms.
- Practice with various differential equations to build intuition and speed.

Conclusion

The Laplace transform to solve initial value problems provides a structured pathway from differential equations to explicit solutions. By transforming the problem into the algebraic domain, solving becomes straightforward, and the inverse transform recovers the solution in the original domain. Mastery of this technique enhances your analytical toolbox for tackling a wide array of problems in engineering, physics, and applied mathematics.

If you want to deepen your understanding, consider practicing with different types of differential equations and initial conditions, or exploring how the Laplace transform handles nonhomogeneous equations and systems.

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