Evaluate The Double Integral_d (x^2 + 2y) DA, D Is Bounded By Y = X, Y = X^3, X 0

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In the realm of multivariable calculus, evaluating double integrals over specified regions is a fundamental skill that allows us to compute areas, volumes, and accumulated quantities. In this article, we will thoroughly analyze and compute the double integral **$\iint_D (x^2 + 2y)\, dA$**, where the region D is bounded by the curves $y = x$, $y = x^3$, and the vertical line $x = 0$. This problem provides an excellent opportunity to explore region boundaries, set up integrals correctly, and employ substitution techniques to simplify calculations.

Understanding the Region D

Identifying the Boundaries

The region D is described by the following boundaries:

- The curve $y = x$ (a straight line passing through the origin with slope 1)
- The curve $y = x^3$ (a cubic curve passing through the origin)
- The vertical boundary $x = 0$ (the y-axis)

These boundaries enclose a region in the xy-plane that lies to the right of the y-axis, between the two curves $y = x^3$ and $y = x$.

Visualizing the Region

To better understand the region D, consider the following points:

- At $x = 0$, both $y = x$ and $y = x^3$ pass through (0, 0).
- For $x > 0$, the line $y = x$ is above the cubic $y = x^3$ because for $x > 0$, $x > x^3$ (since $x^3 < x$ for x in (0,1)).
- The curves intersect at the origin and at $x = 1$, where:

$y = x = x^3$
$\Rightarrow x = x^3$
$\Rightarrow x^3 - x = 0$
$\Rightarrow x(x^2 - 1) = 0$

=> $x = 0$ or $x = 1$

Thus, the two curves intersect at (0, 0) and (1, 1).

Region D Summary

- x ranges from 0 to 1.
- For each fixed x in [0, 1], y varies between $y = x^3$ (lower boundary) and $y = x$ (upper boundary).

Setting Up the Double Integral

Choosing the Order of Integration

Since the region is naturally bounded between $y = x^3$ and $y = x$ for x in [0, 1], it is straightforward to set up the double integral with respect to y first, and then x. The integral becomes:

$$\int_{x=0}^{1} \int_{y=x^3}^{x} (x^2 + 2y)\, dy\, dx$$

Alternatively, we could consider integrating with respect to x first, but given the region's description, the y-integration approach is cleaner.

Explicit Form of the Integral

The double integral over D is:

$$\int_{x=0}^{1} \int_{y=x^3}^{x} (x^2 + 2y)\, dy\, dx$$

This setup respects the boundaries and allows us to evaluate step-by-step.

Calculating the Inner Integral

Inner Integral with Respect to y

Compute:

$$I(x) = \int_{y=x^3}^{x} (x^2 + 2y)\, dy$$

Since x is treated as a constant during integration, the integral becomes:

$$I(x) = \int_{y=x^3}^{x} x^2 \, dy + \int_{y=x^3}^{x} 2y \, dy$$

Calculating each part separately:

1. $\int x^2 \, dy = x^2 y \,|_{y=x^3}^{x} = x^2 (x - x^3) = x^2 (x - x^3)$
2. $\int 2y \, dy = y^2 \,|_{y=x^3}^{x} = x^2 - (x^3)^2 = x^2 - x^6$

Adding both parts:

$$I(x) = x^2 (x - x^3) + x^2 - x^6$$

Simplify:

$$I(x) = x^2 \cdot x - x^2 \cdot x^3 + x^2 - x^6 = x^3 - x^5 + x^2 - x^6$$

Evaluating the Outer Integral

Integral with Respect to x

Now, the original integral reduces to:

$$\int_0^{1} [x^3 - x^5 + x^2 - x^6] \, dx$$

Break into separate integrals:

$$\int_0^1 x^3 \, dx - \int_0^1 x^5 \, dx + \int_0^1 x^2 \, dx - \int_0^1 x^6 \, dx$$

Calculate each:

1. $\int_0^1 x^3 \, dx = (x^4)/4 \,|_0^1 = 1/4$
2. $\int_0^1 x^5 \, dx = (x^6)/6 \,|_0^1 = 1/6$
3. $\int_0^1 x^2 \, dx = (x^3)/3 \,|_0^1 = 1/3$
4. $\int_0^1 x^6 \, dx = (x^7)/7 \,|_0^1 = 1/7$

Putting it all together:

$$\text{Total} = 1/4 - 1/6 + 1/3 - 1/7$$

Combining the Fractions

Find the common denominator, which is 84 (LCM of 4, 6, 3, 7):

- 1/4 = 21/84
- 1/6 = 14/84
- 1/3 = 28/84
- 1/7 = 12/84

Compute:

$$\text{Total} = (21/84) - (14/84) + (28/84) - (12/84) = (21 - 14 + 28 - 12)/84 = (23)/84$$

Final value of the double integral:

$$\iint_D (x^2 + 2y)\, dA = 23/84$$

Conclusion and Summary

Key Takeaways

- The region D is bounded by $y = x$, $y = x^3$, and $x = 0$, with x ranging from 0 to 1.
- Setting up the double integral with y as the inner variable is straightforward due to the natural bounds.
- The integral simplifies nicely by integrating term-by-term.
- The final value of the double integral over the specified region is 23/84.

Practical Implications

This process demonstrates how to approach double integrals over complex regions, emphasizing the importance of:

- Understanding the region's boundaries
- Choosing an appropriate order of integration
- Simplifying the integrand before integration
- Carefully calculating and combining fractional results

Additional Tips for Evaluating Double Integrals

1. **Sketch the Region:** Always draw the region to visualize boundaries and intersections.
2. **Select the Integration Order:** Choose the order that simplifies calculations based on the region's shape.
3. **Set Up Limits Carefully:** Ensure the limits reflect the exact region boundaries.
4. **Handle the Integrand:** Simplify the integrand if possible before integrating.
5. **Compute Step-by-Step:** Break the integral into manageable parts, especially when dealing with polynomial expressions.

Final Thoughts

Evaluating double integrals over regions bounded by curves such as $y = x$ and $y = x^3$ provides valuable experience in multivariable calculus. By carefully analyzing the region, setting up the integral correctly, and methodically performing the calculations, students and practitioners can solve complex integrals with confidence. The specific problem discussed confirms that with proper setup and algebraic simplification, even seemingly complicated integrals can be computed efficiently, leading to precise results like the value of 23/84 for the integral in question.

Frequently Asked Questions

How do I set up the double integral for the region bounded by y = x, y = x^3, and x ≥ 0?

Since the region is bounded by y = x and y = x^3 for x ≥ 0, and noting that y = x^3 is below y = x for x ≥ 0, you can set up the integral with x from 0 to 1 (where the curves intersect), and y from y = x^3 up to y = x. Thus, the integral becomes:

$\int_0^1 \int_{y=x^3}^{y=x} (x^2 + 2y)\, dy\, dx$.

What is the best order of integration for evaluating the double integral over this region?

Integrating with respect to y first, then x, is straightforward because the region's bounds are defined by y between x^3 and x, with x from 0 to 1. So, the order dy dx simplifies the evaluation process.

How do I compute the inner integral ∫_{y=x^3}^{x} (x^2 + 2y) dy?

Treat x as a constant during the inner integral. Integrate term-by-term:

$\int (x^2)\, dy = x^2 y$,
$\int 2y\, dy = y^2$.

Evaluate from $y = x^3$ to $y = x$:

$x^2 x - x^2 x^3 + (x)^2 - (x^3)^2 = x^3 - x^2 x^3 + x^2 - x^6$.

What is the value of the outer integral after computing the inner integral?

After evaluating the inner integral, you'll have an expression in terms of x:

$x^3 - x^2 x^3 + x^2 - x^6$. Simplify this to get a function of x, then integrate from 0 to 1 to find the total value of the double integral.

Are there any special considerations or substitutions needed in this integral?

Since the region and integrand are straightforward polynomials, no substitutions are necessary. Just carefully evaluate the inner integral first, then perform the outer integral. Be mindful of simplifying the algebraic expressions before integrating.

What is the final value of the double integral ∬_D (x^2 + 2y) dA over the given region?

By performing the integrations:

Inner integral: $x^3 - x^2 x^3 + x^2 - x^6$

Outer integral from 0 to 1:

$\int_0^1 [x^3 - x^2 x^3 + x^2 - x^6]\, dx$.

Simplify the integrand and integrate term-by-term to obtain the final result, which evaluates to 1/10.

Additional Resources

Comprehensive Evaluation of the Double Integral \(\iint_D (x^2 + 2y) \, dA\), where \(D\) is bounded by \(y = x\), \(y = x^3\), and \(x \geq 0\)

Introduction

Double integrals are fundamental tools in multivariable calculus, enabling us to compute areas, volumes, and accumulated quantities over two-dimensional regions. The integral in question,

$$
\iint_D (x^2 + 2y) \, dA,
$$

evaluates the total 'weighted' sum of the function $x^2 + 2y$ over a region D. The region D is specified by the boundaries:

- $y = x$,
- $y = x^3$,
- and $x \geq 0$.

This setup offers an insightful case study into how to define, describe, and evaluate double integrals over complex regions bounded by curves. It also provides an opportunity to explore the choice of integration order, the limits of integration, and the calculations involved.

Geometric Description of the Region D

Boundaries and Region Visualization

Understanding the region D geometrically is critical. Let's analyze the bounds:

- Curve $y = x$: A straight line passing through the origin with a slope of 1.
- Curve $y = x^3$: A cubic curve passing through the origin, with a flatter slope near $x=0$ and steeper as x increases.
- Region $x \geq 0$: Focused on the right half-plane.

Intersection Points

To determine the limits, find where the two boundary curves intersect:

$$
x = x^3 \implies x^3 - x = 0 \implies x(x^2 - 1) = 0.
$$

This yields solutions:

$$
x = 0,\quad x = \pm 1.
$$

Since $x \geq 0$, the relevant intersection points are at:

- $x = 0$,
- $x = 1$.

Corresponding y-values are:

- At $x=0$, $y=0$,
- At $x=1$, $y=1$.

Thus, the region D is bounded between $x=0$ and $x=1$, with the curves $y=x^3$ and $y=x$ forming the lower and upper boundaries respectively.

Choosing the Integration Strategy

Deciding the Order of Integration

When setting up a double integral, one must choose whether to integrate with respect to x first (inner integral) or y first (inner integral). The choice depends on the ease of expressing bounds.

- Integrating with respect to y first:

For a fixed x in $[0,1]$:

- Lower boundary: $y = x^3$,
- Upper boundary: $y = x$.

The limits of y are straightforward:

$$
y: x^3 \leq y \leq x,
$$

since $x^3 \leq x$ for $x \in [0,1]$.

- Integrating with respect to x first:

For a fixed y:

- x varies between the inverse of the curves:

$$
y = x^3 \implies x = y^{1/3},
$$
$$
y = x \implies x = y.
$$

- Since $x \geq 0$, and for $0 \leq y \leq 1$:

\[
y^{1/3} \leq x \leq y,
\]

but note that for \(0 \leq y \leq 1\), \(y^{1/3} \leq y\) is true only when \(0 \leq y \leq 1\).

Given the simplicity of expressing bounds, integrating with respect to \(y\) first is often more straightforward, especially because the bounds are explicitly functions of \(x\).

Setting Up the Double Integral

Based on the geometric analysis, the region \(D\) can be described as:

\[
D = \left\{ (x,y) \mid 0 \leq x \leq 1,\quad x^3 \leq y \leq x \right\}.
\]

Thus, the double integral becomes:

\[
\boxed{
\iint_D (x^2 + 2y) \, dA = \int_{x=0}^{1} \int_{y=x^3}^{y=x} (x^2 + 2y) \, dy \, dx.
}
\]

Step-by-Step Evaluation

Step 1: Inner integral with respect to \(y\)

\[
I(x) = \int_{y=x^3}^{x} (x^2 + 2y) \, dy.
\]

Since \(x\) is treated as a constant during differentiation:

\[
I(x) = \int_{x^3}^{x} x^2 \, dy + \int_{x^3}^{x} 2y \, dy.
\]

Calculating each:

- \(\int_{x^3}^{x} x^2 \, dy = x^2 (y) \big|_{x^3}^{x} = x^2 (x - x^3) = x^3 - x^2 \cdot x^3 = x^3 - x^5.\)

- \(\int_{x^3}^{x} 2y \, dy = 2 \cdot \frac{y^2}{2} \big|_{x^3}^{x} = y^2 \big|_{x^3}^{x} = x^2 - (x^3)^2 = x^2 - x^6.\)

Adding these results:

$$
I(x) = (x^3 - x^5) + (x^2 - x^6) = x^2 + x^3 - x^5 - x^6.
$$

Step 2: Outer integral with respect to x

Now, integrate $I(x)$ over x from 0 to 1:

$$
\iint_D (x^2 + 2y) \, dA = \int_0^1 \left(x^2 + x^3 - x^5 - x^6 \right) dx.
$$

Break into separate integrals:

$$
\int_0^1 x^2 \, dx + \int_0^1 x^3 \, dx - \int_0^1 x^5 \, dx - \int_0^1 x^6 \, dx.
$$

Calculate each:

- $\int_0^1 x^2 \, dx = \frac{x^3}{3} \big|_0^1 = \frac{1}{3}.$
- $\int_0^1 x^3 \, dx = \frac{x^4}{4} \big|_0^1 = \frac{1}{4}.$
- $\int_0^1 x^5 \, dx = \frac{x^6}{6} \big|_0^1 = \frac{1}{6}.$
- $\int_0^1 x^6 \, dx = \frac{x^7}{7} \big|_0^1 = \frac{1}{7}.$

Putting it all together:

$$
\boxed{
\iint_D (x^2 + 2y) \, dA = \frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{7}.
}
$$

Final Simplification

Combine the fractions:

$$
\frac{1}{3} + \frac{1}{4} - \frac{1}{6} - \frac{1}{7}.
$$

Find a common denominator. The denominators are 3, 4, 6, and 7. The least common denominator (LCD):

- $LCM(3,4,6,7) = 84$.

Express each fraction with denominator 84:

- \(\frac{1}{3} = \frac{28}{84}\),
- \(\frac{1}{4} = \frac{21}{84}\),
- \(\frac{1}{6} = \frac{14}{84}\),
- \(\frac{1}{7} = \frac{12}{84}\).

Now sum:

\[
\frac{28}{84} + \frac{21}{84} - \frac{14}{84} - \frac{12}{84} = \frac{(28 + 21 - 14 - 12)}{84} = \frac{(49 - 26)}{84} = \frac{23}{84}.
\]

Final Result:

\[
\boxed{
\i

Evaluate The Double Integral D X 2 2y Da D Is Bounded By Y X Y X 3 X 0

Evaluate The Double Integral D X 2 2y Da D Is Bounded By Y X Y X 3 X 0

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