Evaluate Yyye Y 2 Dv, Where E Is The Solid Hemisphere X 2 1 Y 2 1 Z2 < 9, Y > 0.

Evaluate Yyye Y 2 Dv, Where E Is The Solid Hemisphere X 2 1 Y 2 1 Z2 < 9, Y > 0.
In the realm of multivariable calculus and vector analysis, evaluating integrals over complex regions is a fundamental skill. The problem statement involves calculating a volume integral of a specific function over a solid hemisphere, defined by the inequality $E: x^2 + y^2 + z^2 < 9$ with the condition $y > 0$. Such integrals are crucial in physics, engineering, and mathematics, especially when dealing with spherical symmetry, gravitational fields, or electromagnetic potentials. This article aims to provide a comprehensive guide to understanding, setting up, and evaluating the integral $\iiint_E y y^2 \, dv$, where the domain E is the upper hemisphere of radius 3.

Understanding the Problem: Evaluating the Integral over a Hemisphere

1. The Domain E: The Solid Hemisphere

The domain E is given by the inequality:
$$
x^2 + y^2 + z^2 < 9,
$$
with the additional condition $y > 0$. This describes the solid upper hemisphere of radius 3 centered at the origin.

- Radius of Hemisphere: $R = 3$
- Center: Origin (0, 0, 0)
- Domain: The region inside the sphere $x^2 + y^2 + z^2 < 9$ with $y > 0$.

The restriction $y > 0$ limits the domain to half of the sphere, specifically the upper half concerning the y-axis.

Setting Up the Integral

2. The Integrand: $y y^2 = y^3$

The integrand simplifies to y^3. Since the integral involves the volume element dv, the full integral becomes:

$$
\iiint_E y^3 \, dv.
$$

The goal is to evaluate:

$$
I = \iiint_{E} y^3 \, dv.
$$

3. Choosing Coordinates: Spherical Coordinates

Spherical coordinates are best suited for regions defined by spheres. Recall:

$$
x = r \sin \phi \cos \theta,
$$

$$
y = r \sin \phi \sin \theta,
$$

$$
z = r \cos \phi,
$$

where:
- $r \ge 0$,
- $0 \le \phi \le \pi$,
- $0 \le \theta < 2\pi$.

The volume element in spherical coordinates is:

$$
dv = r^2 \sin \phi \, dr \, d\phi \, d\theta.
$$

Domain in spherical coordinates:

- Radius: $0 \le r \le 3$,
- Azimuthal angle: $0 \le \theta \le 2\pi$,

- Polar angle: Since $y > 0$, and $y = r \sin \phi \sin \theta$, the condition $y > 0$ leads to:

$$
r \sin \phi \sin \theta > 0.
$$

Because $r \ge 0$ and $\sin \phi \ge 0$ in $[0, \pi]$, the inequality reduces to:

$$
\sin \theta > 0,
$$

which implies:

$$
0 < \theta < \pi,
$$

since $\sin \theta > 0$ in the first and second quadrants.

Expressing the Integrand in Spherical Coordinates

$$
y = r \sin \phi \sin \theta,
$$

so,

$$
y^3 = (r \sin \phi \sin \theta)^3 = r^3 (\sin \phi)^3 (\sin \theta)^3.
$$

The integral now becomes:

$$
I = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{3} r^3 (\sin \phi)^3 (\sin \theta)^3 \times r^2 \sin \phi \, dr \, d\phi \, d\theta.
$$

Simplifying:

$$
I = \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{3} r^5 (\sin \phi)^4 (\sin \theta)^3 \, dr \, d\phi \, d\theta.
$$

Since the integration limits for θ are from 0 to π, and for r from 0 to 3, the integral is:

$$
I = \int_{0}^{\pi} (\sin \phi)^4 \, d\phi \times \int_{0}^{\pi} (\sin \theta)^3 \, d\theta \times \int_{0}^{3} r^5 \, dr.
$$

Evaluating the Integral Step-by-Step

4. Radial Integral:

$$
\int_{0}^{3} r^5 \, dr = \frac{r^6}{6} \Big|_{0}^{3} = \frac{3^6}{6} = \frac{729}{6} = 121.5.
$$

5. Angular Integrals:

- Integral over ϕ:

$$
\int_{0}^{\pi} (\sin \phi)^4 \, d\phi.
$$

Recall that:

$$
\int_{0}^{\pi} \sin^n \phi \, d\phi = \sqrt{\pi} \frac{\Gamma \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n}{2} + 1 \right)},
$$

or more straightforwardly, for integer powers, known formulas:

$$
\int_{0}^{\pi} \sin^n \phi \, d\phi = \frac{2 \times 3 \times \dots \times (n-1)}{n \times (n-2) \times \dots \times 1} \quad \text{if } n \text{ is even}.
$$

Specifically,

$$
\int_{0}^{\pi} \sin^4 \phi \, d\phi = \frac{3 \pi}{8}.
$$

- Integral over θ:

$$
\int_{0}^{\pi} (\sin \theta)^3 \, d\theta.
$$

Using the standard integral:

$$
\int_{0}^{\pi} \sin^n \theta \, d\theta = \frac{\sqrt{\pi} \, \Gamma \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n}{2} + 1 \right)}.
$$

Alternatively, known result for odd powers:

$$
\int_{0}^{\pi} \sin^3 \theta \, d\theta = \frac{4}{3}.
$$

Final Calculation of the Integral

Putting all parts together:

$$
I = \left(\frac{3 \pi}{8} \right) \times \left(\frac{4}{3} \right) \times 121.5.
$$

Simplify step-by-step:

1. Multiply the angular integrals:

$$
\frac{3 \pi}{8} \times \frac{4}{3} = \frac{3 \pi \times 4}{8 \times 3} = \frac{4 \pi}{8} = \frac{\pi}{2}.
$$

\]

2. Multiply this result by the radial integral:

\[
I = \frac{\pi}{2} \times 121.5 = \frac{\pi}{2} \times \frac{243}{2} = \frac{\pi \times 243}{4}.
\]

Thus,

\[
\boxed{
I = \frac{243 \pi}{4}.
}
\]

Summary and Key Takeaways

- Using spherical coordinates simplifies the evaluation of integrals over spherical regions.
- The symmetry of the region and integrand helps reduce the problem to manageable integrals.
- The condition \(y > 0 \) restricts the azimuthal angle \(\theta \) to \(0 < \theta < \pi \).
- The integral over the radial coordinate is straightforward, with the upper limit being the radius of the hemisphere.
- Known integrals of powers of sine functions are essential tools for evaluating the angular parts.

Additional Considerations

6. Applications of Such Integrals

Evaluating integrals like \(\iiint_E y^3 \, dv \) over hemispherical domains has numerous applications:

- Physics: Calculating mass, charge, or energy distributions in hemispherical objects

Frequently Asked Questions

What is the geometric interpretation of the surface Y yye Y 2 Dv, where E is a solid hemisphere defined by $X^2 + Y^2 + Z^2 < 9$, with $Y > 0$?

The surface Y yye Y 2 Dv appears to be a typo or misformatted; however, if it refers to a surface within the solid hemisphere defined by $X^2 + Y^2 + Z^2 < 9$ with $Y > 0$, it likely relates to a specific boundary or region within the upper hemisphere for evaluation.

What is the significance of the condition $Y > 0$ in the context of the solid hemisphere $X^2 + Y^2 + Z^2 < 9$?

The condition $Y > 0$ restricts the region to the upper half of the solid hemisphere, i.e., the hemisphere above the XY-plane.

How do you evaluate a triple integral over the solid hemisphere of radius 3?

To evaluate a triple integral over the hemisphere, convert to spherical coordinates with appropriate bounds: r from 0 to 3, θ from 0 to 2π, and φ from 0 to $\pi/2$ (since $Y > 0$), then set up and integrate accordingly.

What is the role of the inequality $X^2 + Y^2 + Z^2 < 9$ in setting up the integral?

It defines the boundary of the solid hemisphere of radius 3, indicating the limits for the radial coordinate in spherical coordinates or for the Cartesian bounds within the sphere.

How can the symmetry of the solid hemisphere simplify the evaluation of integrals involving Y?

Since the hemisphere is symmetric about the Z-axis, integrals involving odd functions of Y over the symmetric domain may evaluate to zero, simplifying calculations.

What coordinate system is most suitable for evaluating integrals over a hemisphere?

Spherical coordinates are most suitable because of the spherical symmetry of the region, making integrals easier to compute.

What are the bounds for θ and φ in spherical coordinates for the upper hemisphere where Y > 0?

θ varies from 0 to 2π, and φ varies from 0 to π/2, covering the entire upper hemisphere.

If the integral involves Y, how do you express Y in spherical coordinates?

In spherical coordinates, Y = r sin(φ) sin(θ).

What are common challenges encountered when evaluating integrals over the solid hemisphere with Y > 0?

Challenges include correctly setting bounds for spherical coordinates, handling integrals of functions involving Y, and managing the symmetry to simplify calculations.

Are there specific techniques recommended for evaluating integrals over hemispherical regions defined by inequalities like X^2 + Y^2 + Z^2 < 9?

Yes, converting to spherical coordinates, exploiting symmetry, and using symmetry properties of the integrand are recommended techniques for efficient evaluation.

Additional Resources

Evaluation of the Yyye Y 2 Dv in the Context of a Solid Hemisphere with the Condition \(X^2 + Y^2 + Z^2 < 9,\, Y > 0 \)

Introduction

In the realm of vector calculus and multivariable calculus, the evaluation of integrals over specific regions is a foundational task that appears frequently in physics, engineering, and mathematics. The problem statement — "Evaluate \(Y y y e \, Y\, 2\, Dv \), where \(E \) is the solid hemisphere defined by \(X^2 + Y^2 + Z^2 < 9 \), \(Y > 0 \)" — appears complex at first glance due to its notation and the geometric region involved.

This detailed review aims to interpret, analyze, and evaluate this integral systematically, delving into geometric interpretation, coordinate transformations, integral setup, and calculation strategies, ensuring clarity at each step.

Clarifying the Mathematical Expression

Before proceeding, it’s essential to interpret the original notation correctly, as the provided expression appears somewhat ambiguous:

- Potential notation interpretations:
- The phrase "Yyye Y 2 Dv" could be a misrendered or abbreviated notation.
- "Y y y e" might refer to the integrand involving Y and possibly y variables.
- The "Y 2 Dv" may represent an integral over a volume Dv, possibly denoting dV.

Given the context, a plausible interpretation is:

$$
\boxed{
\text{Evaluate} \quad \iiint_{E} \text{(integrand)} \, dV,
}
$$

where E is the solid hemisphere, and the integrand involves powers of Y (or y), possibly Y^2.

Assumption for the purpose of this review:

- The integral is over the region E, the solid hemisphere $X^2 + Y^2 + Z^2 < 9$, with $Y > 0$.
- The integrand is Y^2, i.e., Y^2.

Therefore, the problem reduces to:

$$
\boxed{
I = \iiint_{E} Y^2 \, dV,
}
$$

where $E = \{ (X,Y,Z) \mid X^2 + Y^2 + Z^2 < 9, \, Y > 0 \}$.

Geometric Description of the Region E

Understanding the region E is foundational for integral evaluation.

The Solid Hemisphere:

- Sphere of radius 3: $X^2 + Y^2 + Z^2 < 9$.
- Hemisphere with $Y > 0$: This indicates the upper or front half of the sphere, depending on the orientation.

Note: Since the inequality involves $Y > 0$, the region is the upper hemisphere in the Y-direction.

Visualization:

- The sphere is centered at the origin with radius 3.
- The hemisphere is the "half" of the sphere where $Y > 0$, i.e., the region in the positive Y-space.

Coordinate System Choice

To evaluate the integral, an appropriate coordinate system simplifies the process:

- Spherical coordinates are most suitable due to the spherical boundary.
- Definition of spherical coordinates:

$$
\begin{cases}
X = r \sin \theta \cos \phi, \\
Y = r \sin \theta \sin \phi, \\
Z = r \cos \theta,
\end{cases}
$$

where

- $r \geq 0$,
- $0 \leq \theta \leq \pi$ (polar angle from the positive Z-axis),
- $0 \leq \phi < 2\pi$ (azimuthal angle from the X-axis in the XY-plane).

Region in Spherical Coordinates

Given the description:

- The sphere: $r \leq 3$.

- Hemisphere $Y > 0$: Since $Y = r \sin \theta \sin \phi$,

$$
Y > 0 \Rightarrow r \sin \theta \sin \phi > 0.
$$

Since $r \geq 0$, the sign depends on $\sin \theta$ and $\sin \phi$.

- $\sin \theta \geq 0$ for $0 \leq \theta \leq \pi$.
- Therefore, the inequality reduces to:

$$
\sin \phi > 0 \quad \text{(since } r, \sin \theta \geq 0 \text{)}.
$$

- $\sin \phi > 0$ for $0 < \phi < \pi$.

Thus, the region E in spherical coordinates:

$$
0 \leq r \leq 3, \quad 0 \leq \theta \leq \pi, \quad 0 < \phi < \pi.
$$

Formulating the Integral

The integrand is assumed to be Y^2. Since $Y = r \sin \theta \sin \phi$, then:

$$
Y^2 = r^2 \sin^2 \theta \sin^2 \phi.
$$

The volume element in spherical coordinates:

$$
dV = r^2 \sin \theta \, dr \, d\theta \, d\phi.
$$

Therefore, the integral becomes:

$$
I = \iiint_{E} Y^2 \, dV = \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{3} r^2 \sin^2 \theta \sin^2 \phi \times r^2
$$

$$
\sin \theta \, dr \, d\theta \, d\phi,
$$

which simplifies to:

$$
I = \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{3} r^4 \sin^3 \theta \sin^2 \phi \, dr \, d\theta \, d\phi.
$$

Step-by-Step Evaluation

Step 1: Integrate over r from 0 to 3

$$
\int_0^3 r^4 \, dr = \left[\frac{r^5}{5} \right]_0^3 = \frac{3^5}{5} = \frac{243}{5}.
$$

Step 2: Integrate over θ from 0 to π

$$
\int_0^\pi \sin^3 \theta \, d\theta.
$$

- Use the reduction formula:

$$
\int \sin^n \theta \, d\theta = - \frac{1}{n} \sin^{n-1} \theta \cos \theta + \frac{n-1}{n} \int \sin^{n-2} \theta \, d\theta.
$$

Alternatively, known integral:

$$
\int_0^\pi \sin^n \theta \, d\theta = \sqrt{\pi} \frac{\Gamma \left(\frac{n+1}{2} \right)}{\Gamma \left(\frac{n}{2} + 1 \right)}.
$$

For $n=3$:

$$
\int_0^\pi \sin^3 \theta \, d\theta = \frac{4}{3}.

\]

Step 3: Integrate over \(\phi \) from 0 to \(\pi \)

\[
\int_0^{\pi} \sin^2 \phi \, d\phi.
\]

- Known integral:

\[
\int_0^{\pi} \sin^2 \phi \, d\phi = \frac{\pi}{2}.
\]

Combining the Results

Putting everything together:

\[
I = \left(\frac{243}{5} \right) \times \left(\frac{4}{3} \right) \times \left(\frac{\pi}{2} \right).
\]

Simplify step by step:

1. Multiply numerator terms:

\[
243 \times 4 \times \pi = 972 \pi.
\]

2. Denominator:

\[
5 \times 3 \times 2 = 30.
\]

Final result:

\[
\boxed{
I = \frac{972 \pi}{30} = \frac{162 \pi}{5}.

}
\]

Summary of the Evaluation Process

- Identified the geometric region as the upper hemisphere of radius 3.
- Chose spherical coordinates for simplicity.
- Determined the bounds for \(r, \theta, \phi \).
- Expressed the integrand \(Y^2 \) in spherical coordinates.
- Calculated the triple integral by successive integrations over \(r, \theta, \phi \).
- Derived the final value \(\boxed{\frac{162 \pi}{5}} \).

Evaluate Yyye Y 2 Dv Where E Is The Solid Hemisphere X 2 1 Y 2 1 Z2 Lt 9 Y Gt 0

Evaluate Yyye Y 2 Dv Where E Is The Solid Hemisphere X 2 1 Y 2 1 Z2 Lt 9 Y Gt 0

Related Articles

- Determine Whether The Two Line Are Parallel, Perpendicular Or Neither.x-3y=6y=1/3x+4
- Find An Equivalent Algebraic Expression For The Composition: Cos(sin()) 14- 2 4+ 2 14+
- In The Hit Show Breaking Bad, What Is The Name Of Jessie Pinkman's Younger Brother?

Back to Home