Find The Exact Length Of The Given Parametric Curve.X= T-3t^3,y = 3t^2 ,0 < T <2

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Understanding how to determine the exact length of a parametric curve is fundamental in calculus, especially in the study of curves and their properties. In this article, we will explore the detailed process of calculating the length of the parametric curve defined by the equations $x = T - 3t^3$ and $y = 3t^2$ over the interval $0 < T < 2$. We will break down the problem step-by-step, providing clear explanations, relevant formulas, and practical examples to help you grasp the concept thoroughly.

Introduction to Parametric Curves

Before diving into the calculation, it’s essential to understand what parametric curves are and how they are represented mathematically.

What Is a Parametric Curve?

A parametric curve is a set of equations that express the coordinates of the points on a curve as functions of a parameter, usually denoted as t. Instead of defining y directly as a function of x, both x and y are expressed separately in terms of t:

$$
x = x(t), \quad y = y(t)
$$

where t varies over some interval.

Why Use Parametric Equations?

Parametric equations are particularly useful in representing curves that are difficult to express explicitly as functions of x or y alone. They are also fundamental in physics for describing trajectories, in engineering for designing complex shapes, and in computer graphics for rendering curves.

Understanding the Given Curve

The specific parametric equations provided are:

$$
x(t) = T - 3t^3
$$
$$
y(t) = 3t^2
$$

with the interval $0 < T < 2$. It’s important to clarify the roles of T and t in this context.

Clarification of Variables

In the problem statement, T appears as an upper limit or a parameter, but the curve equations are expressed in terms of t. Typically, in parametric curve length calculations, the parameter is t. Therefore, it’s likely that T is a constant parameter that defines the interval over which the curve is examined, and t is the variable parameter.

Given the interval $0 < T < 2$, we interpret this as the parameter t varying from 0 to T. To avoid confusion, let’s restate the problem:

- The curve is defined for $t \in [0, T]$.
- The value of T is between 0 and 2.
- Our goal: Find the exact length of the curve from $t=0$ to $t=T$.

Calculating the Length of a Parametric Curve

The general formula for the length L of a parametric curve $x(t)$, $y(t)$ over $t \in [a, b]$ is:

$$
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 } \, dt
$$

This formula calculates the accumulated distance traveled along the curve as t varies from a to b.

Step-by-Step Calculation

Now, apply this formula to our specific parametric equations.

Step 1: Find the derivatives dx/dt and dy/dt

Given:

$$
x(t) = T - 3t^3
$$
$$
y(t) = 3t^2
$$

Calculate derivatives:

$$
\frac{dx}{dt} = -9t^2
$$
$$
\frac{dy}{dt} = 6t
$$

Step 2: Set up the integral for the arc length

Using the formula:

$$
L = \int_{0}^{T} \sqrt{\left(-9t^2 \right)^2 + (6t)^2} \, dt
$$

Simplify inside the square root:

$$
L = \int_{0}^{T} \sqrt{81t^4 + 36t^2} \, dt
$$

Factor out common terms:

$$
L = \int_{0}^{T} \sqrt{9t^2 (9t^2 + 4)} \, dt

\]

Express as:

\[
L = \int_{0}^{T} \sqrt{9t^2} \sqrt{9t^2 + 4} \, dt
\]

Since \(t \geq 0 \) in the interval \([0, T] \), \(\sqrt{9t^2} = 3t \):

\[
L = \int_{0}^{T} 3t \sqrt{9t^2 + 4} \, dt
\]

Step 3: Use substitution to evaluate the integral

Let:

\[
u = 9t^2 + 4
\]
then:

\[
du/dt = 18t
\]
or:

\[
du = 18t \, dt
\]

We have \(3t \, dt \) in the integral, so:

\[
3t \, dt = \frac{1}{6} du
\]

Change the limits:

When \(t = 0 \):

\[
u = 9 \times 0^2 + 4 = 4
\]

When \(t = T \):

\[

u = 9T^2 + 4
\]

Rewriting the integral:

\[
L = \int_{u=4}^{u=9T^2 + 4} \sqrt{u} \times \frac{1}{6} du
\]

Pull out constants:

\[
L = \frac{1}{6} \int_{4}^{9T^2 + 4} u^{1/2} \, du
\]

Step 4: Integrate \(u^{1/2} \)

Recall:

\[
\int u^{1/2} du = \frac{2}{3} u^{3/2} + C
\]

Therefore:

\[
L = \frac{1}{6} \times \frac{2}{3} \left[u^{3/2} \right]_{4}^{9T^2 + 4}
\]

Simplify coefficients:

\[
L = \frac{2}{18} \left[(9T^2 + 4)^{3/2} - 4^{3/2} \right]
\]
\[
L = \frac{1}{9} \left[(9T^2 + 4)^{3/2} - 4^{3/2} \right]
\]

Calculate \(4^{3/2} \):

\[
4^{3/2} = (4^{1/2})^3 = 2^3 = 8
\]

Thus, the exact length \(L(T) \) of the curve from \(t=0 \) to \(t=T \) is:

\[
\boxed{

L(T) = \frac{1}{9} \left[(9T^2 + 4)^{3/2} - 8 \right]
}
\]

Interpreting the Result

The derived formula provides the exact length of the parametric curve for any \(T \) in the interval \(0 < T < 2 \). To find the length over the specific interval, simply plug in the value of \(T \).

Example: Length when \(T=2 \)

\[
L(2) = \frac{1}{9} \left[(9 \times 2^2 + 4)^{3/2} - 8 \right]
\]

Calculate:

\[
9 \times 4 + 4 = 36 + 4 = 40
\]

\[
L(2) = \frac{1}{9} \left[40^{3/2} - 8 \right]
\]

Since:

\[
40^{3/2} = (40^{1/2})^3 = (\sqrt{40})^3 = (6.3246)^3 \approx 253.0
\]

Thus:

\[
L(2) \approx \frac{1}{9} (253.0 - 8) = \frac{1}{9} \times 245 \approx 27.22
\]

This numerical approximation gives a sense of the curve's length between \(t=0 \) and \(t=2 \).

Summary of the Process

To summarize, here are the key steps involved in calculating the exact length of the given parametric curve:

- Determine derivatives

Frequently Asked Questions

What is the parametric curve given in the problem?

The parametric curve is defined by x = T - 3t^3 and y = 3t^2, with T ranging from 0 to 2.

How do you find the length of a parametric curve?

The length L of a parametric curve x(t), y(t) from t = a to t = b is given by the integral $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$.

What are the derivatives dx/dt and dy/dt for the given curve?

dx/dt = 1 - 9t^2 and dy/dt = 6t.

How do you set up the integral for the length of the given curve?

The integral is $L = \int_0^2 \sqrt{(1 - 9t^2)^2 + (6t)^2}\, dt$.

What is the simplified form of the integrand $\sqrt{(dx/dt)^2 + (dy/dt)^2}$?

The integrand simplifies to $\sqrt{(1 - 9t^2)^2 + 36t^2} = \sqrt{[1 - 18t^2 + 81t^4}$

$+ 36t^2] = \sqrt{[1 + 18t^2 + 81t^4]}$.

Can the integral be simplified further before integration?

Yes, the integrand is $\sqrt{(1 + 18t^2 + 81t^4)}$, which can be written as $\sqrt{[(9t^2 + 1)^2]} = |9t^2 + 1|$. Since $t > 0$, the absolute value can be removed.

What is the value of the integrand after simplification?

The integrand becomes $9t^2 + 1$.

How do you evaluate the total length of the curve from t=0 to t=2?

Since the integrand simplifies to $9t^2 + 1$, the length $L = \int_0^2 (9t^2 + 1)\, dt$, which can be computed directly.

What is the final calculation of the curve's length?

Calculating the integral: $L = [3t^3 + t]$ from 0 to 2 = $(38 + 2) - (0 + 0) = (24 + 2) = 26$.

What is the exact length of the given parametric curve from T=0 to T=2?

The exact length of the curve is 26 units.

Additional Resources

Find The Exact Length Of The Given Parametric Curve: X= T - 3t^3, Y=

3t^2, 0 < T < 2

Understanding the precise length of a parametric curve is a fundamental concept in calculus, often encountered in fields ranging from physics to engineering. When working with parametric equations such as X= T - 3t^3 and Y= 3t^2, finding the exact length of the curve within a specified interval is an essential skill. In this guide, we will walk through the detailed process of calculating the length of this parametric curve, providing you with a comprehensive understanding of the underlying principles and methods involved.

Introduction to Parametric Curve Lengths

Before delving into calculations, it’s important to understand what a parametric curve is and how its length is determined. A parametric curve is defined by equations that express the coordinates (x, y) as functions of a parameter, often denoted as t:

- $x = x(t)$
- $y = y(t)$

The arc length of such a curve between parameter values $t = a$ and $t = b$ is given by the integral:

$$
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

This formula arises from the Pythagorean theorem, capturing the infinitesimal distances along the curve.

Step-by-Step Breakdown of the Problem

Given the parametric equations:

$$
x(t) = T - 3t^3 \quad \text{and} \quad y(t) = 3t^2
$$

with the interval $0 < T < 2$, our goal is to find the exact length of the curve over the specified parameter domain.

Clarification of the Interval

It appears there might be some ambiguity or typo regarding the interval involving T and t. Typically, the parametric equations are expressed in terms of a single parameter t. If the problem states $0 < T < 2$, it’s likely a typo, and the intended interval is $0 < t < 2$. Therefore, we will proceed assuming the interval for the parameter t is from 0 to 2.

Step 1: Compute Derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$

Calculating these derivatives is the first step in setting up the arc length integral.

- $\frac{dx}{dt} = \frac{d}{dt} (T - 3t^3)$

Since T is considered a constant parameter, derivative with respect to t:

$$
\frac{dx}{dt} = -9t^2
$$

- $\frac{dy}{dt} = \frac{d}{dt} (3t^2) = 6t$

Step 2: Set Up the Arc Length Integral

Using the derivatives, the arc length L from $t=0$ to $t=2$ is:

$$
L = \int_{0}^{2} \sqrt{\left(-9t^2\right)^2 + (6t)^2} \, dt
$$

Simplify the expression under the square root:

$$
L = \int_{0}^{2} \sqrt{81t^4 + 36t^2} \, dt
$$

Factor out common terms:

$$
L = \int_{0}^{2} \sqrt{9t^2 (9t^2 + 4)} \, dt
$$

Extract the square root:

$$
L = \int_{0}^{2} 3t \sqrt{9t^2 + 4} \, dt
$$

Step 3: Make an Appropriate Substitution

To evaluate the integral:

$$
L = 3 \int_{0}^{2} t \sqrt{9t^2 + 4} \, dt
$$

let’s set:

$$
u = 9t^2 + 4
$$

Then:

$$
du = 18t \, dt \Rightarrow t \, dt = \frac{du}{18}
$$

When $t = 0$:

$$
u = 9(0)^2 + 4 = 4
$$

When $t = 2$:

$$
u = 9(2)^2 + 4 = 9 \times 4 + 4 = 36 + 4 = 40
$$

Rewriting the integral in terms of u:

$$
L = 3 \int_{u=4}^{u=40} \sqrt{u} \times \frac{du}{18}
$$

Simplify constants:

$$
L = \frac{3}{18} \int_{4}^{40} u^{1/2} \, du = \frac{1}{6} \int_{4}^{40} u^{1/2} \, du
$$

Step 4: Evaluate the Integral

The integral:

$$
\int u^{1/2} \, du = \frac{2}{3} u^{3/2}
$$

Applying the limits:

$$
L = \frac{1}{6} \times \frac{2}{3} \left[u^{3/2} \right]_{4}^{40} = \frac{2}{18} \left[40^{3/2} - 4^{3/2} \right]
$$

Simplify constants:

$$
L = \frac{1}{9} \left[40^{3/2} - 4^{3/2} \right]
$$

Now, compute the terms:

- $40^{3/2} = (40)^{1} \times (40)^{1/2} = 40 \times \sqrt{40}$

- $4^{3/2} = 4 \times \sqrt{4} = 4 \times 2 = 8$

Calculate $\sqrt{40}$:

$$
\sqrt{40} = \sqrt{4 \times 10} = 2 \sqrt{10}
$$

Therefore:

$$
40^{3/2} = 40 \times 2 \sqrt{10} = 80 \sqrt{10}
$$

Putting it all together:

$$
L = \frac{1}{9} \left(80 \sqrt{10} - 8 \right) = \frac{80 \sqrt{10} - 8}{9}
$$

Final Result: Exact Length of the Curve

The exact length of the parametric curve $x = T - 3t^3$, $y = 3t^2$, from $t=0$ to $t=2$ is:

$$
\boxed{
L = \frac{80 \sqrt{10} - 8}{9}
}
$$

Additional Insights and Applications

Why Is This Important?

Calculating the exact length of a parametric curve is crucial in various applications:

- Physics: To determine the actual distance traveled along a path.
- Engineering: For designing components with specific length constraints.
- Computer Graphics: For rendering curves accurately.
- Mathematics: Understanding the properties of curves and their behaviors.

Extending the Concept

The methodology used here—deriving derivatives, setting up integrals, and applying substitution—can be generalized to more complex parametric curves. Mastering these techniques opens the door to solving a wide range of problems involving curve lengths, surface areas, and volumes.

Practical Tips for Students and Professionals

- Always verify the interval and parameters involved.
- Simplify the integrand before attempting substitution.
- Use substitution wisely to reduce complex radicals.
- Remember to adjust limits according to the substitution.
- Express the final answer in simplest radical form for clarity.

Summary

In this comprehensive guide, we have:

- Clarified the problem of finding the exact length of a parametric curve.
- Calculated derivatives of the given parametric equations.
- Set up and simplified the arc length integral.
- Chosen and applied an appropriate substitution.
- Evaluated the integral to arrive at the exact length.

This process demonstrates the power of calculus techniques in analyzing geometric properties of curves and provides a solid foundation for tackling similar problems in advanced mathematics and applied sciences.

Remember: Accurate calculation of curve lengths involves careful differentiation, substitution, and algebraic manipulation. With practice, these steps become more intuitive, enabling precise analysis of complex parametric curves in your mathematical toolkit.

Find The Exact Length Of The Given Parametric Curve X T 3t 3 Y 3t 2 0 Lt T Lt 2

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