Find The Natural Frequencies And Mode Shapes Of The System Shown In Fig For M1=m2=1kg

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Understanding the dynamic behavior of mechanical systems is crucial in designing structures and machinery that can withstand vibrations without failure. When analyzing a system's vibrational characteristics, two key concepts are the natural frequencies and the corresponding mode shapes. These are intrinsic properties of a system, indicating the frequencies at which the system tends to oscillate naturally and the deformation patterns associated with these oscillations. In this article, we will explore the process of determining the natural frequencies and mode shapes of a specific two-mass system, where the masses M1 and M2 are both 1 kg, as depicted in the hypothetical figure.

System Description and Assumptions

Overview of the System

Although the figure is not provided here, typical two-mass systems consist of two masses connected by springs and/or dampers. For the purpose of this analysis, we assume the system to be a simplified, idealized model with the following characteristics:

- Two point masses: M1 and M2, each with mass 1 kg.
- Connecting elements: Springs (with stiffnesses k1 and k2) connect the masses to each other and possibly to fixed supports.
- No damping: To focus solely on natural frequencies and mode shapes, damping is neglected.
- Linear, elastic system: The springs obey Hooke’s law, and displacements are small.

Typical Configuration Assumed

A common configuration involves:

- M1 connected to a fixed support via spring k1.
- M2 connected to M1 via spring k2.
- Both masses free to oscillate in one dimension (say, horizontal).

This configuration leads to coupled equations of motion, which can be analyzed to find the system's natural frequencies and mode shapes.

Mathematical Formulation of the System

Deriving Equations of Motion

Let’s denote:

- $x_1(t)$: displacement of mass M1 from equilibrium.
- $x_2(t)$: displacement of mass M2 from equilibrium.

Assuming the springs obey linear elasticity, the forces on each mass are:

- For M1: $-k_1 x_1$ from the fixed support and $-k_2 (x_1 - x_2)$ from the spring connecting M1 and M2.
- For M2: $-k_2 (x_2 - x_1)$.

Applying Newton’s second law:

$$
M_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0
$$

$$
M_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0
$$

Given $M_1 = M_2 = 1\, \text{kg}$, the equations simplify to:

$$
\ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0
$$

$$
\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0
$$

Assuming Harmonic Solutions

To find the natural frequencies, assume solutions of the form:

$$
x_1(t) = X_1 e^{i \omega t}
$$

$$
x_2(t) = X_2 e^{i \omega t}
$$

where ω is the angular frequency to be determined, and X_1, X_2 are the mode shape amplitudes.

Substituting into the equations:

$$
- \omega^2 X_1 + (k_1 + k_2) X_1 - k_2 X_2 = 0
$$

$$
- \omega^2 X_2 + k_2 X_2 - k_2 X_1 = 0
$$

This yields a homogeneous system:

$$
[(k_1 + k_2) - \omega^2] X_1 - k_2 X_2 = 0
$$

$$
- k_2 X_1 + (k_2 - \omega^2) X_2 = 0
$$

Expressed in matrix form:

$$
\begin{bmatrix}
(k_1 + k_2) - \omega^2 & -k_2 \\
- k_2 & k_2 - \omega^2
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix}
= \mathbf{0}
$$

Non-trivial solutions exist only when the determinant of the coefficient matrix is zero:

$$
\det
\begin{bmatrix}
(k_1 + k_2) - \omega^2 & -k_2 \\
- k_2 & k_2 - \omega^2
\end{bmatrix}
= 0
$$

Calculating the determinant:

$$
[(k_1 + k_2) - \omega^2](k_2 - \omega^2) - (k_2)^2 = 0
$$

This characteristic equation allows us to solve for ω^2, the squared natural frequencies.

Calculating the Natural Frequencies

Deriving the Characteristic Equation

Expanding the determinant:

$$
[(k_1 + k_2) - \omega^2](k_2 - \omega^2) - k_2^2 = 0
$$

$$
[(k_1 + k_2)k_2 - (k_1 + k_2)\omega^2 - k_2 \omega^2 + \omega^4] - k_2^2 = 0
$$

Simplify:

$$
(k_1 + k_2)k_2 - (k_1 + 2k_2)\omega^2 + \omega^4 - k_2^2 = 0
$$

Rearranged as:

$$
\omega^4 - (k_1 + 2k_2) \omega^2 + [(k_1 + k_2)k_2 - k_2^2] = 0
$$

Note that:

$$
(k_1 + k_2)k_2 - k_2^2 = k_1 k_2
$$

Thus, the quadratic in ω^2 is:

$$
\omega^4 - (k_1 + 2k_2) \omega^2 + k_1 k_2 = 0
$$

Let $y = \omega^2$:

$$
y^2 - (k_1 + 2k_2) y + k_1 k_2 = 0
$$

The solutions:

$$
y_{1,2} = \frac{(k_1 + 2k_2) \pm \sqrt{(k_1 + 2k_2)^2 - 4 k_1 k_2}}{2}
$$

Corresponding natural frequencies:

$$
\boxed{
\omega_{1,2} = \sqrt{y_{1,2}}
}
$$

Numerical Example: Assigning Spring Constants

Suppose:

- $k_1 = 100\, \text{N/m}$
- $k_2 = 50\, \text{N/m}$

Compute:

$$
(k_1 + 2k_2) = 100 + 100 = 200
$$

$$
\Delta = \sqrt{200^2 - 4 \times 100 \times 50} = \sqrt{40,000 - 20,000} = \sqrt{20,000} \approx 141.42
$$

Then:

$$
y_1 = \frac{200 + 141.42}{2} = \frac{341.42}{2} \approx 170.71
$$

$$
y_2 = \frac{200 - 141.42}{2} = \frac{58.58}{2} \approx 29.29
$$

Thus:

$$
\omega_1 = \sqrt{170.71} \approx 13.06\, \text{rad/sec}
$$

$$
\omega_2 = \sqrt{29.29} \approx 5.41\, \text{rad/sec}
$$

The corresponding natural frequencies in Hz:

$$
f_{1,2} = \frac{\omega_{1,2}}{2\pi} \approx \frac{13.06}{6.283} \approx 2.08\, \text{Hz}

\]
\[
f_{2,2} = \frac{5.41}{6.283} \approx 0.86\, \text{Hz}
\]

Determining Mode Shapes

Eigenvector Computation

For each natural frequency, the mode shape corresponds to the eigenvector \(\mathbf{X} = [X_1, X_2]^T \). To find

Frequently Asked Questions

What is the significance of natural frequencies and mode shapes in a mechanical system?

Natural frequencies are the specific frequencies at which a system tends to vibrate when disturbed, and mode shapes describe the deformation patterns associated with each natural frequency. Understanding these helps in predicting resonances and ensuring the system's structural integrity.

How do the masses M1 and M2 influence the natural frequencies of the system?

The masses M1 and M2 directly affect the system's inertia; increasing mass generally lowers the natural frequencies, while decreasing mass raises them. Equal masses, such as M1 = M2 = 1kg, simplify the analysis and influence the symmetry of the mode shapes.

What method can be used to determine the natural frequencies and mode shapes of the given system?

The typical approach involves setting up the system's equations of motion, deriving the mass and stiffness matrices, and solving the eigenvalue problem (determinant of $[K - \omega^2 M] = 0$). The eigenvalues give the squared natural frequencies, and the eigenvectors correspond to the mode shapes.

Why is it important to analyze mode shapes alongside natural frequencies?

Analyzing mode shapes helps visualize how the system deforms at each natural frequency, aiding in identifying potential areas of stress or failure, and designing damping or control strategies to mitigate resonance effects.

How does the symmetry of the masses (both being 1kg) affect the natural frequencies and mode shapes?

Symmetrical masses lead to symmetric mode shapes and can result in degenerate or closely spaced natural frequencies, simplifying the analysis and often producing predictable vibration patterns.

If the system is modeled as a two-degree-of-freedom system, what are the typical steps to find its natural frequencies and mode shapes?

First, write the equations of motion, then construct the mass (M) and stiffness (K) matrices. Next, solve the eigenvalue problem $[K - \omega^2 M] \varphi = 0$, where eigenvalues ω^2 give the squared natural frequencies, and eigenvectors φ are the mode shapes. Finally, interpret the results to understand vibration behavior.

Additional Resources

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Introduction

The investigation of natural frequencies and mode shapes is fundamental in understanding the dynamic behavior of mechanical systems. These parameters are crucial for predicting resonant conditions, ensuring structural integrity, and optimizing design for vibrational performance. In this article, we delve into the detailed analysis of a specific two-mass system, where both masses, M1 and M2, are given as 1 kg each. Although the figure referenced is not provided here, typical configurations involve masses interconnected by springs and possibly dampers, forming a classic vibrational system.

The goal is to determine the system's natural frequencies and corresponding mode shapes, which describe how the masses move relative to each other during free vibration. This analysis encompasses formulating the equations of motion, deriving the characteristic equations, and solving for eigenvalues and eigenvectors, which represent the natural frequencies and mode shapes, respectively.

System Description and Assumptions

Typical Configuration

While the specific figure is not available, a common two-degree-of-freedom (2-DOF) system under consideration often involves:

- Mass M1 connected to a fixed support via a spring with stiffness \(k_1\).
- Mass M2 connected to M1 via a spring with stiffness \(k_2\).

- Both masses are free to move horizontally or vertically depending on the setup.
- Damping effects are neglected for the purpose of natural frequency analysis, assuming undamped free vibrations.

Assumptions

- The system is linear and elastic.
- The masses are point masses.
- Displacements are small, allowing linearization.
- No damping influences are considered, focusing solely on undamped natural frequencies and mode shapes.
- The springs obey Hooke’s law.

Parameters

- $M_1 = M_2 = 1\, \text{kg}$
- Spring constants k_1, k_2 are assumed known or are to be treated as parameters for the analysis.

Formulation of Equations of Motion

Kinematic Variables

Let:

- $x_1(t)$ be the displacement of mass M_1.
- $x_2(t)$ be the displacement of mass M_2.

Derivation

Applying Newton's second law to each mass yields:

For M_1:

$$
M_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0
$$

For M_2:

$$
M_2 \ddot{x}_2 + k_2 (x_2 - x_1) = 0
$$

Since $M_1 = M_2 = 1\, \text{kg}$, the equations simplify to:

$$
\ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0
$$

\[
\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0
\]

Matrix Formulation and Eigenvalue Problem

Expressing the equations in matrix form:

\[
\mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = 0
\]

where

\[
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

and

\[
\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}
\]

Seeking solutions of the form:

\[
\mathbf{x}(t) = \boldsymbol{\phi} e^{i \omega t}
\]

leads to the generalized eigenvalue problem:

\[
(\mathbf{K} - \omega^2 \mathbf{M}) \boldsymbol{\phi} = 0
\]

Non-trivial solutions exist when:

\[
\det (\mathbf{K} - \omega^2 \mathbf{M}) = 0
\]

which simplifies to:

\[

$$
\det \left(\begin{bmatrix} k_1 + k_2 - \omega^2 & -k_2 \\ -k_2 & k_2 - \omega^2 \end{bmatrix} \right) = 0
$$

Calculating the determinant:

$$
(k_1 + k_2 - \omega^2)(k_2 - \omega^2) - (-k_2)(-k_2) = 0
$$

$$
(k_1 + k_2 - \omega^2)(k_2 - \omega^2) - k_2^2 = 0
$$

Expanding:

$$
(k_1 + k_2)(k_2 - \omega^2) - \omega^2(k_2 - \omega^2) - k_2^2 = 0
$$

Further expansion:

$$
(k_1 + k_2)k_2 - (k_1 + k_2)\omega^2 - \omega^2 k_2 + \omega^4 - k_2^2 = 0
$$

Rearranged as:

$$
\omega^4 - \omega^2 [(k_1 + k_2) + k_2] + (k_1 + k_2)k_2 - k_2^2 = 0
$$

Simplify:

$$
\omega^4 - \omega^2 (k_1 + 2k_2) + k_2 k_1 = 0
$$

This quadratic in ω^2 yields the natural frequencies:

$$
\boxed{
\omega^2_{1,2} = \frac{(k_1 + 2k_2) \pm \sqrt{(k_1 + 2k_2)^2 - 4 k_1 k_2}}{2}
}
$$

Determination of Natural Frequencies

Parameter Selection

For explicit calculations, specific values of k_1 and k_2 are necessary. Let's assume:

- $k_1 = 100\, \text{N/m}$
- $k_2 = 50\, \text{N/m}$

Insert into the formula:

$$
\omega^2_{1,2} = \frac{(100 + 100) \pm \sqrt{(100 + 100)^2 - 4 \times 100 \times 50}}{2}
$$

Compute:

$$
\omega^2_{1,2} = \frac{200 \pm \sqrt{200^2 - 20000}}{2}
$$

$$
= \frac{200 \pm \sqrt{40000 - 20000}}{2} = \frac{200 \pm \sqrt{20000}}{2}
$$

$$
\sqrt{20000} \approx 141.42
$$

Thus:

$$
\omega_1^2 = \frac{200 + 141.42}{2} = \frac{341.42}{2} \approx 170.71
$$
$$
\omega_2^2 = \frac{200 - 141.42}{2} = \frac{58.58}{2} \approx 29.29
$$

The natural frequencies are:

$$
\boxed{
\omega_1 \approx \sqrt{170.71} \approx 13.07\, \text{rad/sec}
}
$$
$$
\omega_2 \approx \sqrt{29.29} \approx 5.41\, \text{rad/sec}
$$

Corresponding frequencies in Hz:

$$
f_{1,2} = \frac{\omega_{1,2}}{2\pi} \approx \frac{13.07}{6.283} \approx 2.08\, \text{Hz}
$$
\[

f_2 \approx 0.86\, \text{Hz}
\]

Mode Shapes Analysis

Eigenvector Calculation

For each \(\omega^2\), solve:

\[
(\mathbf{K} - \omega^2 \mathbf{M}) \boldsymbol{\phi} = 0
\]

Using the first eigenvalue \(\omega_1^2 \approx 170.71\):

\[
\begin{bmatrix}
k_1 + k_2 - \omega_1^2 & -k_2 \\
-k_2 & k_2 - \omega_1^2
\end{bmatrix}
\boldsymbol{\phi} = 0
\]

Substituting the known values:

\[
\begin{bmatrix}
100 + 50 - 170.71 & -50 \\
-50 & 50 - 170.71
\end{bmatrix}
\boldsymbol{\phi} = 0
\]

\[
\begin{bmatrix}
-20.71 & -50 \\
-50 & -120.71
\end{bmatrix

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