Find The Point On The Plane X 2y + 3z = 6 That Is Closest To The Point (0, 1, 1).

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Introduction

Determining the closest point on a plane to a given point in space is a classic problem in analytical geometry. This problem has practical significance in fields such as computer graphics, physics, and engineering, where minimizing distances between points and surfaces is a common task. The challenge involves finding the point on the given plane that minimizes the Euclidean distance to the specified point, which in this case is (0, 1, 1).

In this article, we will thoroughly explore the methods to solve this problem, including geometric interpretations and algebraic procedures, culminating in a step-by-step solution that finds the closest point explicitly.

Understanding the Problem

The Given Data

- Plane Equation: $x + 2y + 3z = 6$
- Point in Space: $P(0, 1, 1)$

The goal is to find the point $Q(x, y, z)$ on the plane such that the distance between P and Q is minimized.

Geometric Approach to the Problem

Distance from a Point to a Plane

The shortest distance from a point to a plane is along the line perpendicular to the plane, which is in the direction of the plane's normal vector.

- Normal vector of the plane: $\vec{n} = (1, 2, 3)$

The closest point Q on the plane to P can be found by projecting P onto the plane along the direction of $\vec{n}$.

Visualizing the Problem

Imagine standing at point P. To reach the closest point Q on the plane, you would move directly along the line perpendicular to the plane, which is aligned with the normal vector $\vec{n}$.

Mathematical Solution

Step 1: Write the Equation of the Line Through P Perpendicular to the Plane

The line passing through $P(0, 1, 1)$ and perpendicular to the plane has parametric equations:

$$
\begin{cases}
x = 0 + t \cdot 1 = t \\
y = 1 + t \cdot 2 = 1 + 2t \\
z = 1 + t \cdot 3 = 1 + 3t
\end{cases}
$$

where t is a parameter representing the distance along the normal vector $\vec{n}$.

Step 2: Find the Intersection of this Line with the Plane

Substitute the parametric equations into the plane equation:

$$
x + 2y + 3z = 6
$$

$$
t + 2(1 + 2t) + 3(1 + 3t) = 6
$$

Simplify:

$$
t + 2 + 4t + 3 + 9t = 6
$$

$$
t + 4t + 9t + (2 + 3) = 6
$$

$$
14t + 5 = 6
$$

Solve for t:

$$
14t = 6 - 5 = 1
$$

$$
t = \frac{1}{14}
$$

Step 3: Find the Coordinates of the Closest Point Q

Plug $t = \frac{1}{14}$ into the parametric equations:

$$
x = t = \frac{1}{14}
$$

$$
y = 1 + 2t = 1 + 2 \times \frac{1}{14} = 1 + \frac{2}{14} = 1 + \frac{1}{7} = \frac{8}{7}
$$

$$
z = 1 + 3t = 1 + 3 \times \frac{1}{14} = 1 + \frac{3}{14} = \frac{14}{14} + \frac{3}{14} = \frac{17}{14}
$$

Therefore, the closest point Q on the plane to $P(0, 1, 1)$ is:

$$
Q\left(\frac{1}{14}, \frac{8}{7}, \frac{17}{14} \right)
$$

Verification and Additional Insights

Verification of the Point on the Plane

Check if Q satisfies the plane equation:

$$
x + 2y + 3z = 6
$$

$$
\frac{1}{14} + 2 \times \frac{8}{7} + 3 \times \frac{17}{14}
$$

Calculate each term:

$$
\frac{1}{14} + \frac{16}{7} + \frac{51}{14}
$$

Express all with denominator 14:

$$
\frac{1}{14} + \frac{32}{14} + \frac{51}{14} = \frac{1 + 32 + 51}{14} = \frac{84}{14} = 6
$$

Confirmed, the point lies on the plane.

Distance from \(P \) to \(Q \)

To find the shortest distance, compute \(\| P - Q \| \):

\[

\sqrt{ \left(0 - \frac{1}{14}\right)^2 + \left(1 - \frac{8}{7}\right)^2 + \left(1 - \frac{17}{14}\right)^2 }

\]

Simplify each component:

\[

\left(- \frac{1}{14} \right)^2 = \frac{1}{196}

\]

\[

1 - \frac{8}{7} = \frac{7}{7} - \frac{8}{7} = - \frac{1}{7}

\]

\[

\left(- \frac{1}{7} \right)^2 = \frac{1}{49}

\]

\[

1 - \frac{17}{14} = \frac{14}{14} - \frac{17}{14} = - \frac{3}{14}

\]

\[

\left(- \frac{3}{14} \right)^2 = \frac{9}{196}

\]

Sum:

\[

\frac{1}{196} + \frac{1}{49} + \frac{9}{196}

\]

Express all with denominator 196:

\[

\frac{1}{196} + \frac{4}{196} + \frac{9}{196} = \frac{14}{196} = \frac{1}{14}

\]

Distance:

\[

\sqrt{\frac{1}{14}} = \frac{1}{\sqrt{14}}

\]

Conclusion:

The minimal distance from \(P \) to the plane is \(\frac{1}{\sqrt{14}} \), and the point \(Q\left(\frac{1}{14}, \frac{8}{7}, \frac{17}{14} \right) \) is the unique closest point.

Alternative Methods

Using the Projection Formula

Another approach involves projecting \(P \) onto the plane using the projection formula:

\[

Q = P - \frac{\text{distance from } P \text{ to the plane}}{\|\vec{n}\|^2} \vec{n}

\]

where:

\[

\text{distance} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}

\]

Given the point \(P(0,1,1) \) and plane \(x + 2y + 3z = 6 \), the calculation confirms the earlier result.

Summary

The problem of finding the closest point on a plane to a given point is elegantly addressed by leveraging the geometry of perpendicular lines and vector projections. By constructing the line through the point \(P \) in the direction of the plane's normal vector, we identified the intersection point as the closest point \(Q \). The calculations show that:

\[

\boxed{

Q = \left(\frac{1}{14}, \frac{8}{7}, \frac{17}{14} \right)

}

\]

and the minimal distance is \(\frac{1}{\sqrt{14}} \).

This method illustrates the power of combining geometric intuition with algebraic techniques, providing a clear and concise solution to the problem.

Frequently Asked Questions

How do I find the point on the plane 2y + 3z = 6 closest to the point (0, 1, 1)?

You can find the closest point by minimizing the distance from (0, 1, 1) to any point (x, y, z) on the plane, often using the projection method or Lagrange multipliers.

What is the geometric approach to find the closest point on a plane to a given point?

Geometrically, the closest point on the plane to an external point is the foot of the perpendicular dropped from the point to the plane.

How do I set up the equations to find the foot of the perpendicular from (0, 1, 1) to the plane 2y + 3z = 6?

You set up the parametric form of the line passing through (0, 1, 1) in the direction of the plane's normal vector, then find the intersection point with the plane by solving for the parameter.

What is the normal vector of the plane 2y + 3z = 6, and how is it used in this problem?

The normal vector is (0, 2, 3). It is used to define the direction of the perpendicular from the point to the plane.

Can you provide the step-by-step solution to find the closest point on the plane to (0, 1, 1)?

Yes. First, write the point and plane equation, then find the perpendicular line from the point in the direction of the normal vector, and solve for the intersection point, which gives the closest point.

What is the final answer for the point on the plane 2y + 3z = 6 closest to (0, 1, 1)?

The closest point is (0, 5/13, 12/13).

Additional Resources

Find The Point On The Plane X 2y + 3z = 6 That Is Closest To The Point (0, 1, 1)

In the realm of analytical geometry, the problem of finding the closest point on a plane to a given external point is both fundamental and instructive. It encapsulates core concepts such as distance measurement, orthogonal projections, and the application of vector calculus. Specifically, this problem involves identifying the point on the plane defined by the equation X + 2y + 3z = 6 that minimizes the Euclidean distance to the external point (0, 1, 1). This task not only sharpens one's understanding of geometric relationships but also demonstrates the practical use of algebraic and calculus-based techniques to solve spatial problems.

In this article, we will explore this problem comprehensively, beginning with the foundational concepts, then moving through the step-by-step solution process, and finally analyzing the implications and applications of these methods in broader contexts.

Understanding the Problem: Geometrical Foundations

The Geometry of Planes and Points

A plane in three-dimensional space can be described mathematically as a flat, two-dimensional surface extending infinitely in all directions, characterized by an equation of the form:

$$ Ax + By + Cz = D $$

where A, B, C are the coefficients that define the orientation of the plane, and D determines its position relative to the origin.

The point in question, $(0, 1, 1)$, lies outside this plane. The goal is to find the point $P = (x, y, z)$ on the plane that is closest to this external point. This point is known as the orthogonal projection of the external point onto the plane.

Distance From a Point to a Plane

The Euclidean distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by:

$$ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} $$

When one point lies on a plane, and the other is external, the shortest distance between them is along the line perpendicular to the plane—i.e., along the plane's normal vector. Therefore, the problem reduces to finding the foot of the perpendicular from the external point onto the plane.

Mathematical Approach to Solving the Problem

Step 1: Identify the Plane and External Point

The plane is given by:

\[x + 2y + 3z = 6 \]

The external point is:

\[P_0 = (0, 1, 1) \]

Step 2: Find the Normal Vector of the Plane

The plane's normal vector \(\vec{n} \) is derived from the coefficients of \(x, y, z \):

\[\vec{n} = (A, B, C) = (1, 2, 3) \]

This vector is perpendicular to the plane and points in the direction of the shortest path from the external point to the plane.

Step 3: Equation of the Line Through the External Point Perpendicular to the Plane

The foot of the perpendicular, say \(P = (x, y, z) \), lies along the line passing through \(P_0 \) in the direction of the normal vector:

\[\text{Line parametric equations:} \]

\[

\begin{cases}

x = x_0 + tA = 0 + t \times 1 = t \\

y = y_0 + tB = 1 + t \times 2 = 1 + 2t \\

z = z_0 + tC = 1 + t \times 3 = 1 + 3t

\end{cases}

\]

where \(t \) is a scalar parameter.

Step 4: Find the Intersection of the Line with the Plane

To find the point \(P \) on the line that lies on the plane, substitute the parametric coordinates into the plane equation:

\[x + 2y + 3z = 6 \]

Substituting:

\[t + 2(1 + 2t) + 3(1 + 3t) = 6 \]

Simplify:

\[

t + 2 + 4t + 3 + 9t = 6

\]

Combine like terms:

\[

(t + 4t + 9t) + (2 + 3) = 6

\]

\[

$$
14t + 5 = 6
$$

Solve for t:

$$
14t = 6 - 5 = 1
$$

$$
t = \frac{1}{14}
$$

Calculating the Closest Point

Step 5: Determine the Coordinates of the Closest Point

Plug the value of $t = \frac{1}{14}$ back into the parametric equations:

$$
x = t = \frac{1}{14}
$$

$$
y = 1 + 2t = 1 + 2 \times \frac{1}{14} = 1 + \frac{2}{14} = 1 + \frac{1}{7} = \frac{8}{7}
$$

$$
z = 1 + 3t = 1 + 3 \times \frac{1}{14} = 1 + \frac{3}{14} = \frac{14}{14} + \frac{3}{14} = \frac{17}{14}
$$

Therefore, the point on the plane closest to \((0, 1, 1) \) is:

\[
\boxed{
\left(\frac{1}{14}, \frac{8}{7}, \frac{17}{14} \right)
}
\]

Verifying the Solution

Verification involves confirming that this point indeed lies on the plane and that the line connecting the external point to this point is perpendicular to the plane.

Check if the point lies on the plane:

\[x + 2y + 3z = \frac{1}{14} + 2 \times \frac{8}{7} + 3 \times \frac{17}{14} \]

Calculate each term:

\[
\frac{1}{14} + 2 \times \frac{8}{7} + 3 \times \frac{17}{14} = \frac{1}{14} + \frac{16}{7} + \frac{51}{14}
\]

Express all with denominator 14:

\[
\frac{1}{14} + \frac{16 \times 2}{14} + \frac{51}{14} = \frac{1}{14} + \frac{32}{14} + \frac{51}{14} = \frac{1 + 32 + 51}{14} = \frac{84}{14} = 6

\]

Since the sum equals 6, the point lies on the plane, confirming the correctness of the solution.

Computing the Distance Between the External Point and the Closest Point

The final step is to determine the shortest distance, which is the Euclidean distance between \((0, 1, 1) \) and the closest point:

\[
d = \sqrt{\left(\frac{1}{14} - 0 \right)^2 + \left(\frac{8}{7} - 1 \right)^2 + \left(\frac{17}{14} - 1 \right)^2}
\]

Expressing the differences:

\[
\frac{1}{14} - 0 = \frac{1}{14}
\]
\[
\frac{8}{7} - 1 = \frac{8}{7} - \frac{7}{7} = \frac{1}{7}
\]
\[
\frac{17}{14} - 1 = \frac{17}{14} - \frac{14}{14} = \frac{3}{14}
\]

Calculate the squares:

\[

\left(\frac{1}{14} \right)^2 = \frac{1}{196}

\]

\[

\left(\frac{1}{7} \right)^2 = \frac{1}{49}

\]

\[

\left(\frac{3}{14} \right)^2 = \frac{9}{196}

\]

Sum:

\[

\frac{1}{196} + \frac{1}{49} + \frac{9}{196}

\]

Express all with denominator 196:

\[

\frac{1}{196} + \frac{4}{196} + \frac{9}{196} = \frac{1 + 4 + 9}{196} = \frac{14}{196} = \frac{1}{14}

\]

Finally, take the square root:

\[

d =

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