Find The Smallest Number Such That When Its Divided 14 , 12, 8 Has A Remainder Of 3

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Understanding problems involving division and remainders is a fundamental aspect of number theory and mathematics. Such problems often appear in various contexts, ranging from simple puzzles to complex algorithms. One particular problem that captures the interest of many students and enthusiasts is determining the smallest number that, when divided by certain numbers, leaves a specific remainder. This article explores the problem: Find the smallest number such that when divided by 14, 12, and 8, it has a remainder of 3, and provides a comprehensive guide on how to approach, solve, and understand this type of problem.

Understanding the Problem: Restating the Question

Before delving into the solution, it’s essential to understand what the problem is asking:

- Find the smallest positive number N such that:
- When N is divided by 14, the remainder is 3.
- When N is divided by 12, the remainder is 3.
- When N is divided by 8, the remainder is 3.

In mathematical terms, this can be expressed as:

$$
\begin{cases}
N \equiv 3 \pmod{14} \\
N \equiv 3 \pmod{12} \\
N \equiv 3 \pmod{8}
\end{cases}
$$

The goal is to find the smallest positive integer N satisfying all these congruences simultaneously.

Breaking Down the Problem: Modular Arithmetic Basics

What is Modular Arithmetic?

Modular arithmetic involves calculations with remainders. When we say:

$$
N \equiv r \pmod{m}
$$

it means that when N is divided by m, the remainder is r. For example, $17 \equiv 3 \pmod{7}$ because dividing 17 by 7 leaves a remainder of 3.

Applying Modular Arithmetic to Our Problem

In our case, each condition states that N leaves a remainder of 3 when divided by 14, 12, and 8 respectively. So, N can be expressed as:

$$
N = k_1 \times 14 + 3
$$
$$
N = k_2 \times 12 + 3
$$
$$
N = k_3 \times 8 + 3
$$

for some integers k_1, k_2, and k_3.

Strategies to Find the Smallest Number

To find the smallest such number, we need to reconcile these three conditions. The process involves:

- Recognizing that $N - 3$ is divisible by 14, 12, and 8.
- Finding the least common multiple (LCM) of 14, 12, and 8.
- Using the LCM to determine the smallest N.

Step 1: Simplify the Conditions

Since $N \equiv 3 \pmod{14}$, $N - 3$ is divisible by 14.

Similarly:

$$
N - 3 \equiv 0 \pmod{14}
$$
$$
N - 3 \equiv 0 \pmod{12}
$$
$$
N - 3 \equiv 0 \pmod{8}
$$

Therefore, $N - 3$ must be a common multiple of 14, 12, and 8.

Step 2: Find the LCM of 14, 12, and 8

The least common multiple of these numbers will give us the smallest positive number divisible by all three.

Let's find the prime factorizations:

- $14 = 2 \times 7$
- $12 = 2^2 \times 3$
- $8 = 2^3$

The LCM takes the highest powers of all primes involved:

- Highest power of 2: 2^3 (from 8)
- Prime 3: 3^1 (from 12)
- Prime 7: 7^1 (from 14)

Thus,

$$
\text{LCM} = 2^3 \times 3 \times 7 = 8 \times 3 \times 7 = 8 \times 21 = 168
$$

So, the smallest number divisible by 14, 12, and 8 is 168.

Calculating the Smallest Number N

Since $N - 3$ must be divisible by 168, the smallest positive $N - 3$ satisfying this is 168 itself. Therefore,

$$
N - 3 = 168
$$
$$
N = 168 + 3 = 171
$$

This is the smallest number that, when divided by 14, 12, and 8, leaves a remainder of 3.

Verifying the Solution

Let's verify that $N = 171$ satisfies all the conditions:

- $171 \div 14 = 12$ remainder 3 ✓
- $171 \div 12 = 14$ remainder 3 ✓
- $171 \div 8 = 21$ remainder 3 ✓

Since all conditions are satisfied, 171 is indeed the smallest such number.

Generalized Approach to Similar Problems

This problem exemplifies a common approach to solving systems of congruences:

1. Express the problem in modular form.
2. Identify the common divisibility conditions.
3. Calculate the LCM of the divisors to find the fundamental period.
4. Adjust for the specific remainders to find the minimal solution.

This approach can be extended to more complex systems involving different remainders and more divisors.

Additional Tips and Common Pitfalls

- **Always verify your solution:** After computing the candidate number, check all conditions to confirm correctness.

- **Remember the importance of prime factorization:** It simplifies finding the LCM and understanding common multiples.

- **Be cautious with negative remainders:** Remainders are typically non-negative and less than the divisor.

- **Use systematic methods:** The Chinese Remainder Theorem (CRT) provides a formal framework for solving systems of simultaneous congruences, especially when the divisors are coprime.

Using the Chinese Remainder Theorem (CRT)

While the above method is straightforward for this particular problem, the Chinese Remainder Theorem (CRT) offers a powerful approach for more complex systems. It states that if the divisors are pairwise coprime, then there exists a unique solution modulo the product of the divisors.

In our example, the divisors are not pairwise coprime (since 12 and 8 share factors), but since we are looking for a common solution where \(N - 3 \) is divisible by all three, the approach of finding the LCM remains valid.

Conclusion: Final Answer and Summary

The smallest number \(N \) such that it leaves a remainder of 3 when divided by 14, 12, and 8 is:

\[
\boxed{171}
\]

This solution was derived by understanding the problem through modular arithmetic, calculating the least common multiple of the divisors, and adjusting for the common remainder. The key steps involved:

- Restating the problem in modular form.
- Recognizing that $N - 3$ must be divisible by all three divisors.
- Computing the LCM of 14, 12, and 8, which is 168.
- Adding 3 to find the smallest N, resulting in 171.

This problem illustrates fundamental concepts in number theory and modular arithmetic, providing a template for approaching similar problems involving remainders and divisibility conditions.

Further Reading and Resources

- Number Theory Textbooks: For a deeper understanding of modular arithmetic and the Chinese Remainder Theorem.
- Online Math Resources: Websites like Khan Academy and Brilliant offer interactive lessons on modular arithmetic.
- Mathematical Software: Tools like WolframAlpha or SageMath can verify calculations and solve modular systems efficiently.

By mastering these concepts, you'll be well-equipped to tackle a wide range of problems involving divisibility, remainders, and advanced number theory techniques.

Frequently Asked Questions

What is the problem asking for in finding the smallest number with specific remainders?

It asks to find the smallest number that, when divided by 14, 12, and 8, leaves a remainder of 3 in each case.

How do you set up the problem mathematically?

You express the number as N, and set up the equations: $N \equiv 3 \pmod{14}$, $N \equiv 3 \pmod{12}$, and $N \equiv 3 \pmod{8}$.

What is the significance of subtracting 3 from the remainders in solving this problem?

Subtracting 3 simplifies the problem to finding a number divisible by 14, 12, and 8, since $N - 3$ is divisible by all three.

How do you find the smallest number that satisfies all the conditions?

Calculate the least common multiple (LCM) of 14, 12, and 8, then add 3 to the LCM to find the smallest number.

What is the least common multiple (LCM) of 14, 12, and 8?

The LCM of 14, 12, and 8 is 336.

What is the final answer to the problem?

The smallest number is 336 + 3 = 339.

Why does adding 3 to the LCM give the smallest number satisfying the conditions?

Because N - 3 is divisible by all three divisors, ensuring N leaves a remainder of 3 when divided by each, and adding 3 adjusts back to the original problem.

Can this method be generalized to find numbers with different remainders?

Yes, by adjusting the calculations to account for the specific remainders, this approach can be adapted to various similar problems.

Additional Resources

Find The Smallest Number Such That When It Is Divided By 14, 12, and 8, It Has a Remainder of 3

Understanding the intricacies of divisibility and remainders is a fundamental aspect of number theory, a branch of mathematics that explores properties of integers. Among the myriad problems in this domain, one classic challenge involves finding the smallest number that satisfies a set of modular conditions. Specifically, the problem at hand asks for the smallest number that, when divided by 14, 12, and 8, leaves a remainder of 3 in each case. This type of problem is not only theoretically intriguing but also has practical applications in areas such as cryptography, computer science, and scheduling algorithms.

In this comprehensive article, we delve into the methodology of solving such problems, exploring concepts like modular arithmetic, the Chinese Remainder Theorem, and systematic problem-solving techniques. We will analyze the problem step-by-step, providing clear explanations and detailed reasoning to ensure a thorough understanding of the process. Whether you're a student, educator, or enthusiast interested in number theory, this exploration offers valuable insights into solving complex divisibility puzzles.

Understanding the Problem Statement

What Is Being Asked?

The core question is: What is the smallest positive number that, when divided by 14, 12, and 8, leaves a remainder of 3 in each case?

Mathematically, this can be expressed as a set of congruences:

- $N \equiv 3 \ (\text{mod } 14)$
- $N \equiv 3 \ (\text{mod } 12)$
- $N \equiv 3 \ (\text{mod } 8)$

Here, N is the number we seek. The problem is to find the minimal positive N satisfying all three conditions simultaneously.

Why Is This Important?

At first glance, the problem seems straightforward: find a number that, when divided by each divisor, leaves a specific remainder. However, the challenge lies in satisfying multiple modular conditions simultaneously—this is where the concept of systems of congruences and the Chinese Remainder Theorem (CRT) come into play. These tools are essential for solving complex divisibility problems efficiently, especially when dealing with multiple moduli that are not necessarily coprime.

Fundamental Concepts in Modular Arithmetic

Before tackling the problem directly, it's crucial to understand the foundational ideas involved in modular arithmetic, which deals with integers and their remainders upon division.

Modular Arithmetic and Congruences

- Congruence relation: Two integers a and b are said to be congruent modulo n if they leave the same remainder when divided by n. This is written as:

$$
a \equiv b \ (\text{mod } n)
$$

- Implication: If $a \equiv b \ (\text{mod } n)$, then $a - b$ is divisible by n.

In our problem, the conditions are expressed as congruences, which facilitate systematic solving.

The Chinese Remainder Theorem (CRT)

The Chinese Remainder Theorem is a powerful result that provides a way to solve systems of simultaneous congruences with pairwise coprime moduli. Although in this problem, the moduli are not necessarily coprime (since 8 and 12 share common factors), the theorem's principles can be adapted or combined with other techniques.

The CRT states that:

- If \(n_1, n_2, ..., n_k \) are pairwise coprime integers, then the system:

\[
\begin{cases}
N \equiv a_1 \ (\text{mod } n_1) \\
N \equiv a_2 \ (\text{mod } n_2) \\
\vdots \\
N \equiv a_k \ (\text{mod } n_k)
\end{cases}
\]

has a unique solution modulo \(N = n_1 \times n_2 \times \cdots \times n_k \).

In our case, since the moduli are not coprime, we need to consider the least common multiple (LCM) and compatibility conditions.

Step-by-Step Solution Approach

Step 1: Express the Conditions Mathematically

Given the problem:

- \(N \equiv 3 \ (\text{mod } 14) \)
- \(N \equiv 3 \ (\text{mod } 12) \)
- \(N \equiv 3 \ (\text{mod } 8) \)

This means:

\[
N - 3 \equiv 0 \ (\text{mod } 14), \quad N - 3 \equiv 0 \ (\text{mod } 12), \quad N - 3 \equiv 0 \ (\text{mod } 8)
\]

or equivalently, \(N - 3 \) is divisible by all three moduli.

Step 2: Find the Common Divisibility Condition

Since \(N - 3 \) is divisible by 14, 12, and 8 simultaneously, the problem reduces to:

\[
N - 3 \equiv 0 \ (\text{mod } \text{lcm}(14, 12, 8))
\]

where lcm denotes the least common multiple.

Calculating the LCM:

- Prime factorization:

- $14 = 2 \times 7$
- $12 = 2^2 \times 3$
- $8 = 2^3$

- Take the highest powers of each prime:

- 2^3 (from 8)
- 3 (from 12)
- 7 (from 14)

- Therefore:

$$
\text{lcm}(14, 12, 8) = 2^3 \times 3 \times 7 = 8 \times 3 \times 7 = 168
$$

Hence,

$$
N - 3 \equiv 0 \ (\text{mod } 168)
$$

which implies:

$$
N = 168k + 3
$$

for some integer $k \geq 0$.

Step 3: Find the Smallest Positive Solution

Since $N = 168k + 3$, the smallest positive N corresponds to $k = 0$:

$$
N = 168 \times 0 + 3 = 3
$$

But, check whether this N satisfies the original conditions:

- $3 \div 14$ leaves a remainder of 3? Yes, since 3 mod 14 is 3.
- $3 \div 12$ leaves a remainder of 3? Yes, 3 mod 12 is 3.
- $3 \div 8$ leaves a remainder of 3? Yes, 3 mod 8 is 3.

All conditions are satisfied by $N = 3$.

Validating the Solution

The initial solution $N = 3$ appears to satisfy all the modular conditions. However, it’s essential to verify if this is indeed the smallest positive number that fulfills the criteria.

- Since the general solution is $N = 168k + 3$, for $k = 0$, $N = 3$.
- For $k = 1$, $N = 168 + 3 = 171$, which is larger.

Thus, the smallest positive number that satisfies the conditions is 3.

Conclusion: The minimal number that, when divided by 14, 12, and 8, leaves a remainder of 3 in each case is 3.

Additional Considerations and Edge Cases

When the Moduli Are Not Pairwise Coprime

In more complex scenarios where the moduli share common factors, the solution process involves checking the compatibility of the congruences:

- For the system:

$$
\begin{cases}
N \equiv a_1 \ (\text{mod } n_1) \\
N \equiv a_2 \ (\text{mod } n_2)
\end{cases}
$$

Compatibility requires that:

$$
a_1 \equiv a_2 \ (\text{mod } \gcd(n_1, n_2))
$$

If this condition isn't met, no solution exists. When it is met, solutions can be found using extended Euclidean algorithms or by reducing the system into a single congruence.

Practical Applications

Understanding this problem has real-world applications:

- Cryptography: Modular arithmetic underpins encryption algorithms like RSA, where systems of congruences are solved to generate keys.
- Scheduling and Planning: Finding periodic events that align after certain cycles.
- Computer Science: Hashing functions, error detection, and correction algorithms often rely on modular computations.

Final Thoughts and Summary

This problem exemplifies the elegance and utility of modular arithmetic and number theory in solving seemingly complex divisibility puzzles. The key steps involved:

1. Expressing conditions as modular congruences.
2. Calculating the least common

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