Find The Unique Solution Of The Initial Value Problemv' = 10 - 3v - V^2, V(0) = 3

Find The Unique Solution Of The Initial Value Problemv' = 10 - 3v - V^2, V(0) = 3 is a fundamental task in differential equations, offering insights into the behavior of dynamic systems modeled by nonlinear ordinary differential equations (ODEs). Understanding how to approach such problems is essential for students, mathematicians, engineers, and scientists who deal with real-world phenomena where initial conditions determine the evolution of a system. In this comprehensive guide, we will explore the process of solving this initial value problem (IVP), analyze its solutions, and discuss the significance of initial conditions in determining unique solutions.

Understanding the Initial Value Problem (IVP)

An initial value problem involves a differential equation coupled with specified initial conditions. For our case, the differential equation is:

$$v' = 10 - 3v - V^2,$$

with the initial condition:

$$V(0) = 3.$$

This problem is nonlinear because of the V^2 term, making its solution approach more intricate than linear equations. The goal is to find a function $V(t)$ satisfying both the differential equation and the initial condition.

Key Elements of the IVP

- Differential Equation: $v' = 10 - 3v - V^2$
- Initial Condition: $V(0) = 3$
- Type: Nonlinear first-order ODE
- Objective: Find the unique solution $V(t)$ that satisfies both the differential equation and initial condition

Methods for Solving Nonlinear First-Order ODEs

Solving nonlinear ODEs often requires specialized methods. For the given problem, possible approaches include:

1. Separation of Variables: When the differential equation can be expressed as $f(V) dV = g(t) dt$.
2. Substitution Methods: Using substitutions to turn the nonlinear equation into a linear one.
3. Qualitative Analysis: Analyzing the equilibrium points and stability.
4. Numerical Methods: When an explicit solution isn't feasible, methods like Euler’s or Runge-Kutta provide approximate solutions.

In our case, the differential equation resembles a Bernoulli equation, which can be transformed into a linear form.

Identifying the Type of Differential Equation

The differential equation:

$$V' = 10 - 3V - V^2$$

can be rewritten as:

$$\frac{dV}{dt} = -V^2 - 3V + 10$$

which suggests it is a Riccati differential equation, a special nonlinear form:

$$\frac{dV}{dt} = a(t) + b(t) V + c(t) V^2$$

In our case, $a(t) = 10$, $b(t) = -3$, and $c(t) = -1$.

Transforming the Riccati Equation into a Second-Order Linear ODE

Riccati equations can often be solved by substitution, converting them into second-order linear equations. The standard substitution is:

$$V(t) = -\frac{1}{u(t)} \frac{du(t)}{dt}$$

which transforms the Riccati equation into a linear second-order differential equation for $u(t)$.

Step-by-Step Solution Process

1. Write the Riccati Equation:

$$\frac{dV}{dt} = 10 - 3V - V^2$$

2. Substitution:

Let:

$$V(t) = -\frac{u'(t)}{u(t)}$$

where $u(t)$ is a new function to be determined.

3. Derive the second-order linear ODE:

Differentiating $V(t)$:

$$
V' = -\frac{u'' u - u'^2}{u^2}
$$

Substitute into the original equation:

$$
-\frac{u'' u - u'^2}{u^2} = 10 - 3 \left(-\frac{u'}{u}\right) - \left(-\frac{u'}{u}\right)^2
$$

Simplify RHS:

$$
10 + \frac{3 u'}{u} - \frac{u'^2}{u^2}
$$

Multiply both sides by u^2:

$$
-(u'' u - u'^2) = 10 u^2 + 3 u' u - u'^2
$$

which simplifies to:

$$
- u'' u + u'^2 = 10 u^2 + 3 u' u - u'^2
$$

Bring all to one side:

$$
- u'' u + u'^2 - 10 u^2 - 3 u' u + u'^2 = 0
$$

Combine like terms:

$$
- u'' u + 2 u'^2 - 10 u^2 - 3 u' u = 0
$$

Divide through by u:

$$
- u'' + 2 \frac{u'^2}{u} - 10 u - 3 u' = 0
$$

This is a complex second-order nonlinear differential equation. Alternatively, recognizing the Riccati structure, it's often easier to find particular solutions directly or to use substitution methods specific to Riccati equations.

4. Finding a Particular Solution:

Since the differential equation is autonomous (no explicit t dependence), look for constant solutions $V(t) = V_p$:

$$
0 = 10 - 3V_p - V_p^2
$$

which is a quadratic in V_p:

$$
V_p^2 + 3V_p - 10 = 0
$$

Solve:

$$
V_p = \frac{-3 \pm \sqrt{9 - 4 \times 1 \times (-10)}}{2} = \frac{-3 \pm \sqrt{9 + 40}}{2} = \frac{-3 \pm \sqrt{49}}{2}
$$

$$
V_p = \frac{-3 \pm 7}{2}
$$

Thus, two particular solutions:

- $V_p = \frac{-3 + 7}{2} = 2$
- $V_p = \frac{-3 - 7}{2} = -5$

5. Use the particular solution $V = 2$ to find the general solution:

Define:

$$
V(t) = 2 + y(t)
$$

Substitute into the original differential equation:

$$
V' = 10 - 3V - V^2
$$

becomes:

$$
y' = 10 - 3(2 + y) - (2 + y)^2
$$

Calculate RHS:

$$
10 - 6 - 3y - (4 + 4 y + y^2) = (10 - 6 - 4) - 3 y - 4 y - y^2 = 0 - 7 y - y^2
$$

So,

$$
y' = - y^2 - 7 y
$$

which is separable:

$$
\frac{dy}{dt} = - y (y + 7)
$$

Separate variables:

$$
\frac{dy}{y (y + 7)} = - dt
$$

Partial fraction decomposition:

$$
\frac{1}{y (y + 7)} = \frac{A}{y} + \frac{B}{ y + 7 }
$$

Solve:

$$
1 = A (y + 7) + B y
$$

Set $y=0$:

$$
1 = A \times 7 \Rightarrow A = \frac{1}{7}
$$

Set $y = -7$:

$$
1 = B \times (-7) \Rightarrow B = - \frac{1}{7}
$$

Thus:

$$
\frac{dy}{ y (y + 7)} = \frac{1}{7} \left(\frac{1}{y} - \frac{1}{ y + 7 } \right) dy
$$

Integrate:

$$
\int \left(\frac{1}{7} \left(\frac{1}{y} - \frac{1}{ y + 7 } \right) \right) dy = - \int dt
$$

$$
\frac{1}{7} (\ln | y | - \ln | y + 7 |) = - t + C
$$

Simplify:

$$
\frac{1}{7} \ln \left| \frac{ y }{ y + 7 } \right| = - t + C
$$

Multiply through by 7:

$$
\ln \left| \frac{ y }{ y + 7 } \right| = -7 t + 7 C
$$

Set $K = e^{7 C}$, absorbing the constant:

$$
\left| \frac{ y }{ y + 7 } \right| = K e^{ -7 t }
$$

Recall $V(t) = 2 + y(t)$, so:

$$
y(t) = V(t) -
$$

Frequently Asked Questions

What is the initial value problem given in the differential equation?

The initial value problem is $dv/dt = 10 - 3v - v^2$ with the initial condition $v(0) = 3$.

How can we rewrite the differential equation to make it more manageable?

We can write it as $dv/dt = -(v^2 + 3v - 10)$ to facilitate partial fraction decomposition or substitution methods.

What method is suitable for solving this first-order nonlinear differential equation?

Separation of variables combined with partial fraction decomposition is suitable, or recognizing it as a Bernoulli-type equation if applicable.

How do we find the equilibrium solutions of the differential equation?

Set $dv/dt = 0$, so solve $10 - 3v - v^2 = 0$ for v, which gives the equilibrium points.

What are the equilibrium solutions of the differential equation?

Solve $v^2 + 3v - 10 = 0$; the solutions are $v = 2$ and $v = -5$.

Given the initial condition v(0) = 3, which equilibrium point does the solution approach?

Since $v(0) = 3$, which is close to $v = 2$, the solution tends toward the equilibrium $v = 2$ over time.

How do we solve the differential equation explicitly?

Perform partial fraction decomposition on $1/(10 - 3v - v^2)$, then integrate both sides with respect to t.

What is the general form of the solution after integration?

The solution involves logarithmic functions resulting from partial fractions, leading to an implicit relation between v and t.

What is the unique solution satisfying v(0) = 3?

By applying the initial condition to the implicit solution, we determine the specific constant, giving the unique solution for v(t).

Why is the solution unique for this initial value problem?

Because the differential equation is continuous and satisfies the conditions of the Picard-Lindelöf theorem, ensuring a unique solution passing through the initial point.

Additional Resources

Find The Unique Solution Of The Initial Value Problem: $v' = 10 - 3v - v^2$, $V(0) = 3$

Introduction

In the realm of differential equations, initial value problems (IVPs) serve as fundamental tools for modeling a myriad of real-world phenomena, from physics and biology to engineering and economics. The problem at hand—finding the unique solution of the initial value problem $v' = 10 - 3v - v^2$, $V(0) = 3$—presents an intriguing case that combines nonlinear dynamics with classical solution techniques. This article meticulously explores the solution process, the mathematical underpinnings, and the implications of the solution's behavior, all within a comprehensive investigative framework suited for scholars and practitioners alike.

Overview of the Differential Equation

The given initial value problem is expressed as:

$$ v' = 10 - 3v - v^2 \quad \text{with} \quad v(0) = 3 $$

The equation is a first-order nonlinear ordinary differential equation (ODE). Its structure suggests it may be classified as a Riccati-type equation, given the quadratic term in v. Recognizing the nature of the equation is crucial because it influences the choice of solution methods and the interpretation of the solution.

Mathematical Foundations and Existence of a Unique Solution

Before delving into solving the differential equation, it is vital to establish whether a unique solution exists for the initial value problem, a fundamental question grounded in the Picard-Lindelöf theorem (also known as the Cauchy-Lipschitz theorem).

Conditions for Existence and Uniqueness

- The function $f(v) = 10 - 3v - v^2$ is continuous and differentiable with respect to v over the entire real line.
- The partial derivative $\partial f / \partial v = -3 - 2v$ is also continuous.

Given these properties, and considering the initial value $v(0) = 3$, the Picard-Lindelöf theorem assures the existence and uniqueness of a solution in some interval around $t=0$.

Solution Strategy

The differential equation's structure suggests that separable variables can be employed to find an explicit solution. The process involves rewriting the equation to isolate v and t, then integrating both sides.

Step 1: Rewrite the Equation

The equation:

$$v' = 10 - 3v - v^2$$

can be expressed as:

$$\frac{dv}{dt} = 10 - 3v - v^2$$

which is separable as:

$$\frac{dv}{10 - 3v - v^2} = dt$$

Step 2: Factor or Complete the Square in the Denominator

To integrate the left side, analyze the quadratic in the denominator:

$$v^2 + 3v - 10$$

Rearranged as:

$$v^2 + 3v - 10$$

Complete the square:

$$v^2 + 3v = v^2 + 3v + \frac{9}{4} - \frac{9}{4} = \left(v + \frac{3}{2}\right)^2 - \frac{9}{4}$$

Thus,

$$v^2 + 3v - 10 = \left(v + \frac{3}{2}\right)^2 - \frac{9}{4} - 10 = \left(v + \frac{3}{2}\right)^2 - \frac{9}{4} - \frac{40}{4} = \left(v + \frac{3}{2}\right)^2 -$$

\frac{49}{4} \]

Therefore, the denominator becomes:

\[10 - 3v - v^2 = - (v^2 + 3v -10) = - \left[\left(v + \frac{3}{2}\right)^2 - \frac{49}{4}\right] \]

which simplifies to:

\[10 - 3v - v^2 = \frac{49}{4} - \left(v + \frac{3}{2}\right)^2 \]

Integration and Explicit Solution

The integral now reduces to:

\[\int \frac{dv}{\frac{49}{4} - \left(v + \frac{3}{2}\right)^2} = \int dt \]

This is a standard integral resembling the form:

\[\int \frac{dx}{a^2 - x^2} \]

which evaluates to:

\[\frac{1}{2a} \ln \left| \frac{a + x}{a - x} \right| + C \]

Step 3: Substitution

Let:

\[u = v + \frac{3}{2} \]

Then:

\[du = dv \]

The integral becomes:

\[\int \frac{du}{\frac{49}{4} - u^2} \]

Set:

\[a = \frac{7}{2} \quad \text{since} \quad a^2 = \frac{49}{4} \]

Thus,

\[\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a + u}{a - u} \right| + C \]

Back to original variables:

$$ \int \frac{dv}{10 - 3v - v^2} = \frac{1}{2a} \ln \left| \frac{a + u}{a - u} \right| + C = \frac{1}{2 \times \frac{7}{2}} \ln \left| \frac{\frac{7}{2} + u}{\frac{7}{2} - u} \right| + C $$

Simplify:

$$ \frac{1}{7} \ln \left| \frac{\frac{7}{2} + v + \frac{3}{2}}{\frac{7}{2} - v - \frac{3}{2}} \right| + C $$

Note that:

$$ \frac{7}{2} + v + \frac{3}{2} = \frac{7 + 3}{2} + v = 5 + v $$

$$ \frac{7}{2} - v - \frac{3}{2} = \frac{7 - 3}{2} - v = 2 - v $$

Therefore, the integral simplifies to:

$$ \frac{1}{7} \ln \left| \frac{v + 5}{2 - v} \right| + C = t + t_0 $$

where t_0 is the constant of integration, which we can set to zero for simplicity.

Explicit Solution Expression

Solving for v yields:

$$ \left| \frac{v + 5}{2 - v} \right| = e^{7(t + C)} $$

Let:

$$ K = e^{7C} $$

then:

$$ \left| \frac{v + 5}{2 - v} \right| = K e^{7t} $$

Considering the initial condition $v(0) = 3$:

$$ \left| \frac{3 + 5}{2 - 3} \right| = K e^{0} \Rightarrow \left| \frac{8}{-1} \right| = K \Rightarrow 8 = K $$

Thus,

$$ \left| \frac{v + 5}{2 - v} \right| = 8 e^{7t} $$

Now, determine the sign:

At $t=0$, $v=3$:

$$ \frac{3 + 5}{2 - 3} = \frac{8}{-1} = -8 $$

Absolute value:

$$ 8 = 8 e^{0} \Rightarrow \text{consistent} $$

So, the general solution is:

$$ \frac{v + 5}{2 - v} = -8 e^{7t} $$

or, considering the sign:

$$ \frac{v + 5}{2 - v} = \pm 8 e^{7t} $$

but since initial condition gives the negative sign, we take:

$$ \frac{v + 5}{2 - v} = -8 e^{7t} $$

Solving for $v(t)$

Cross-multiplied:

$$ v + 5 = -8 e^{7t} (2 - v) $$

$$ v + 5 = -16 e^{7t} + 8 v e^{7t} $$

Bring all v-terms to one side:

$$ v - 8 v e^{7t} = -16 e^{7t} - 5 $$

Factor v:

$$ v (1 - 8 e^{7t}) = -16 e^{7t} - 5 $$

Finally, solve for $v(t)$:

$$ v(t) = \frac{-16 e^{7t} - 5}{1 - 8 e^{7t}} $$

This provides the explicit solution satisfying the initial condition $v(0) = 3$.

Analysis of the Solution

The

Find The Unique Solution Of The Initial Value Problemv 10 3v

V 2 V 0 3

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